now general $x_n \to x$:

$\Rightarrow \quad x^{-1} x_n \to 1$

$\overset{\text{above}}{\Rightarrow} \quad (x^{-1} x_n)^{-1} \to 1 \quad \Rightarrow \quad x_n^{-1} \longrightarrow x^{-1}$

$$\underset{x_n^{-1} \cdot x}{\overset{\shortparallel}{(x^{-1}x_n)^{-1}}}$$

$\square$

<u>12.8. Lemma</u>: Let A be Banach algebra, $a \in A$ and $\lambda, \mu \in \rho(a)$. Then

i) $R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$, i.e.

$$\frac{1}{\lambda - a} - \frac{1}{\mu - a} = (\mu - \lambda) \frac{1}{\lambda - a} \cdot \frac{1}{\mu - a}$$

resolvent equation

ii)
$$\lim_{\mu \to \lambda} \frac{1}{\lambda - \mu} \left(R(\lambda) - R(\mu) \right) = - R(\lambda)^2$$

<u>Proof</u>: i) $R(\lambda) = R(\lambda)\,(\mu - a)\, R(\mu)$

$= R(\lambda)\,((\lambda - a) + (\mu - \lambda))\, R(\mu)$

$= R(\mu) + R(\lambda)(\mu - \lambda) R(\mu)$

ii) $\dfrac{1}{\lambda - \mu} (R(\lambda) - R(\mu)) \overset{i)}{=} - R(\lambda) R(\mu) \xrightarrow{\mu \to \lambda} - R(\lambda)^2$

$\square$

12.9. Theorem: Let A be Banach algebra and $a \in A$. Then $\sigma(a)$ is compact and non-empty and

$$\sigma(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$$

12.10. Remarks: 1) Main statement is

$$\sigma(a) \neq \emptyset$$

Note that for $A = M_n(\mathbb{C})$ this means: every matrix T has at least one eigenvalue, i.e., the equation $\det(\lambda 1 - T) = 0$ has at least one solution

→ fundamental theorem of algebra

2) Of course, for this it is crucial to work over $\mathbb{C}$. For $F = \mathbb{R}$, it can happen that the spectrum is empty.

e.g.: $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ has $\sigma(a) = \{+i, -i\}$

but for $a \in M_2(\mathbb{R})$ the corresponding spectrum is empty!

$\sigma_{\mathbb{R}}(a) \subset \mathbb{R}$

Proof of 12.9.: 12.7. implies

$\lambda - a$ invertible $\Rightarrow$ $\mu - a$ invertible if

$|\lambda - \mu|$ sufficiently small

$\Rightarrow \varrho(a)$ open

$\Rightarrow \sigma(a) = \mathbb{C} \setminus \varrho(a)$ closed

Now consider $|\lambda| > \|a\|$

$$\Rightarrow (\lambda \cdot 1 - a)^{-1} = \frac{1}{\lambda} \underbrace{\left(1 - \frac{a}{\lambda}\right)^{-1}}_{\substack{\text{exists, by 12.6., since} \\ \|\frac{a}{\lambda}\| = \frac{\|a\|}{|\lambda|} < 1}}$$

$\Rightarrow \lambda \in \varrho(a)$

$\Rightarrow \sigma(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$

thus: $\sigma(a)$ is closed and bounded,

i.e., $\sigma(a)$ compact

remains to show: $\sigma(a) \neq \emptyset$

Assume $\sigma(a) = \emptyset$, i.e., $\varrho(a) = \mathbb{C}$

$\Rightarrow z \longmapsto R(z) = \frac{1}{z - a}$ is defined on $\mathbb{C}$

[idea: $R(z)$ "holomorphic", $\lim_{z \to \infty} R(z) = 0 \overset{\text{Liouville!?}}{\Longrightarrow} R(z) = 0$]

We have to go from operator-valued function $R(z)$ to a scalar-valued one by applying linear functional

$0 \in \varrho(a) \Rightarrow a^{-1}$ exists and $a^{-1} \neq 0$

$\xRightarrow[\text{Banach}]{\text{Hahn}} \exists L \in A^* \text{ s.th. } L(a^{-1}) \neq 0$

Define now $f: \mathbb{C} \to \mathbb{C}$ by

$$f(z) := L(R(z)) = L\left(\frac{1}{z-a}\right)$$

then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= L\left(\lim_{\Delta z \to 0} \frac{R(z+\Delta z) - R(z)}{\Delta z}\right)$$

$$= L(-R(z)^2) \quad \text{by 12.8 (ii)}$$

$\Rightarrow f'(z)$ exists for all $z \in \mathbb{C}$, thus

f holomorphic on $\mathbb{C}$, i.e.

f entire function

furthermore: $\lim_{z \to \infty} f(z) = 0$, since we have for $|z| > \|a\|$:

$$\|R(z)\| = \left\| \frac{1}{z-a} \right\|$$

$$= \frac{1}{|z|} \left\| \frac{1}{1-\frac{a}{|z|}} \right\|$$

$$\overset{12.6}{\leq} \frac{1}{|z|} \cdot \frac{1}{1-\frac{\|a\|}{|z|}} = \frac{1}{|z|-\|a\|}$$

$$\xrightarrow{z \to \infty} 0$$

thus $R(z) \xrightarrow{z \to \infty} 0$

$\Rightarrow \underbrace{L(R(z))}_{f(z)} \longrightarrow L(0) = 0$ since $L \in A^*$ bounded

thus: $\lim\limits_{z \to \infty} f(z) = 0$

$\Rightarrow$ f bounded on $\mathbb{C}$, and entire

$\overset{\text{Liouville}}{\Longrightarrow}$ f constant $= 0$ $\Rightarrow f \equiv 0$

but then: $f(0) = L(R(0)) = L(a^{-1}) \neq 0$

contradiction

$\Rightarrow \sigma(a) \neq \emptyset$

$\square$

(12-14

12.10. Corollary (Theorem of Gelfand-Mazur):

Let A be a Banach algebra in which each element $a \neq 0$ is invertible. Then $A \cong \mathbb{C}$.

Proof: $a \in A \overset{12.9.}{\Longrightarrow} \sigma(a) \neq \emptyset$, i.e.

$\exists \lambda \in \sigma(a)$: $\lambda 1 - a$ not invertible

$\Rightarrow \lambda 1 - a = 0$

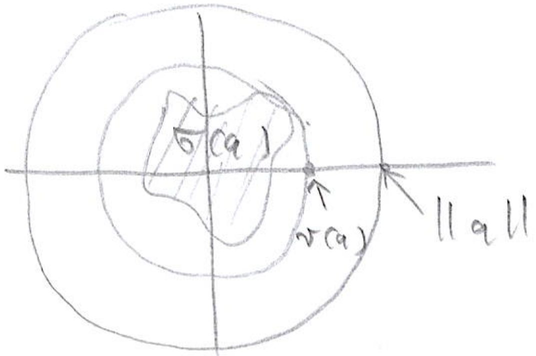
$\Rightarrow \lambda \cdot 1 = a$ $\square$

12.11. Remark: This implies that the quaternions cannot be equipped with a norm that makes it to a Banach algebra.

12.12. Def.: Let A be Banach algebra and $a \in A$. Then

$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$

is the <u>spectral radius</u> of a.



12.13. Theorem (spectral radius formula):

1) Let A be a Banach algebra and $a \in A$. Then

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

2) If A is a C^*-algebra and $a \in A$ normal (i.e., $aa^* = a^*a$), then

$$r(a) = \|a\|$$

12.14. Remarks: 1) Note: $r(a)$ is defined in terms of the algebra structure of A, whereas the right hand depends on the norm / topology.

2) In general it can happen that $r(a) < \|a\|$; for example: $a \neq 0$ with $\sigma(a) = \{0\}$

$$\Rightarrow r(a) = 0, \text{ but } \|a\| \neq 0$$

e.g. $a =$ Volterra operator or

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Proof of 12.13.: 1) note: $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$, since

$$\lambda^n - a^n = (\lambda - a)(\lambda^{n-1} + \lambda^{n-2} a + \dots + \lambda a^{n-2} + a^{n-1})$$