12.13. Theorem (spectral radius formula):

1) Let A be a Banach algebra and $a \in A$. Then

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

2) If A is a C^*-algebra and $a \in A$ normal (i.e., $aa^* = a^*a$), then

$$r(a) = \|a\|$$

12.14. Remarks: 1) Note: $r(a)$ is defined in terms of the algebra structure of A, whereas the right hand depends on the norm/topology.

2) In general it can happen that $r(a) < \|a\|$; for example: $a \neq 0$ with $\sigma(a) = \{0\}$

$$\Rightarrow r(a) = 0, \text{ but } \|a\| \neq 0$$

e.g. $a =$ Volterra operator or

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Proof of 12.13.: 1) note: $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$, since

$$\lambda^n - a^n = (\lambda - a)(\lambda^{n-1} + \lambda^{n-2} a + \dots + \lambda a^{n-2} + a^{n-1})$$

thus: $\lambda^n - a^n$ invertible $\Rightarrow$ $\lambda - a$ invertible with

$$(\lambda - a)^{-1} = (\lambda^{n-1} + \dots + a^{n-1})(\lambda^n - a^n)^{-1}$$

hence:

$$\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n) \overset{12.1.}{\Longrightarrow} |\lambda|^n \leq \|a^n\| \quad \forall n$$

$$\Rightarrow |\lambda| \leq \liminf \|a^n\|^{1/n} \qquad \forall \lambda \in \sigma(a)$$

$$\Rightarrow r(a) \leq \liminf \|a^n\|^{1/n}$$

remains to show: $r(a) \geq \limsup \|a^n\|^{1/n}$

Idea: $R(z) = \frac{1}{z-a} = \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$ if $\|a\| < |z|$

power series expansion of $R(z)$ is valid for all circles where $R(z)$ is holomorph, i.e. for $|z| > r(a)$

classical formula for radius of convergence of power series $\sum \frac{\alpha_n}{z^{n+1}}$ is

$$\limsup |\alpha_n|^{1/n} \;?\; \limsup \|a^n\|^{1/n}$$

Problem: $R(z)$ is operator-valued, we have to reduce it to $\mathbb{C}$, by applying linear functional

Let $L \in A^*$, then put

$$f(z) := L(R(z)) = L\left(\frac{1}{z-a}\right)$$

$\Rightarrow$ $f(z) : \rho(a) \to \mathbb{C}$ holomorph and

$$f(z) = \sum_{n=0}^{\infty} \frac{L(a^n)}{z^{n+1}} \quad \text{for } |z| > \|a\|$$

$\Rightarrow$ this power series expansion is also valid for all $|z| > r(a)$, thus

$$\limsup |L(a^n)|^{1/n} \leq r(a)$$

Consider now $r > r(a)$, then

$$|L(a^n)|^{1/n} \leq r \quad \text{for } n \text{ sufficiently large}$$

$$\Rightarrow \frac{|L(a^n)|}{r^n} \leq 1 \qquad \text{—— " ——}$$

$$\Rightarrow \sup_{n \in \mathbb{N}} \frac{|L(a^n)|}{r^n} < \infty \qquad \forall L \in A^*$$

$\xRightarrow[\text{7.6.}]{\text{Uniform Boundedness}}$

$$\sup_{n \in \mathbb{N}} \left\| \frac{a^n}{r^n} \right\| < \infty$$

$$\sup_{x \in M} |L(x)| < \infty \ \forall L \in A^* \Rightarrow \sup_{x \in M} \|x\| < \infty$$

$$\qquad \text{"} \qquad\qquad\qquad\qquad \text{"}$$

$$\sup_{\hat{x} \in \hat{M}} |\hat{x}(L)| < \infty \ \forall L \in A^* \Rightarrow \sup_{\hat{x} \in \hat{M}} \|\hat{x}\| < \infty$$

$$\Rightarrow \exists\, C' : \left\| \frac{a^n}{r^n} \right\| \leq C' \quad \forall n$$

i.e., $\quad \|a^n\| \leq C' r^n$

$$\Rightarrow \quad \|a^n\|^{1/n} \leq C'^{1/n} \cdot r \xrightarrow{n \to \infty} r$$

$$\Rightarrow \limsup \|a^n\|^{1/n} \leq r \quad \forall r > r(a)$$

$$\Rightarrow \limsup \|a^n\|^{1/n} \leq r(a)$$

and thus:

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

2) Let A be C^*-algebra, and $aa^* = a^* a$

$$\Rightarrow \|a^2\|^2 = \|(a^2)^* a^2\|$$

$$= \| a^* \underbrace{a^* a}_{aa^*} a \|$$

$$= \|(a^* a)^* (a^* a)\|$$

$$= \|a^* a\|^2$$

$$= \|a\|^4$$

$$\Rightarrow \|a^2\| = \|a\|^2$$

$\overset{\text{induction}}{\Longrightarrow} \quad \|a^{2^n}\| = \|a\|^{2^n}$ (note: a normal $\Rightarrow a^k$ normal)

thus: $$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = a$$ $\square$

Consider now $a \in B \subset A$

A, B Banach algebras

In general, $\sigma(a)$ will depend on the algebra, i.e. $\sigma_A(a) \subset \sigma_B(a)$

but "$\neq$" possible

12.15. Example: Let

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ and

$\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$

Put $A := C(\mathbb{T})$

$B := \overline{\{p : \mathbb{T} \to \mathbb{C} \mid p \text{ polynom in } z\}}^{\|\cdot\|}$

note: $B \subsetneq A = \overline{\{p : \mathbb{T} \to \mathbb{C} \mid p \text{ polynom in } z \text{ and } \bar{z}\}}^{\|\cdot\|}$

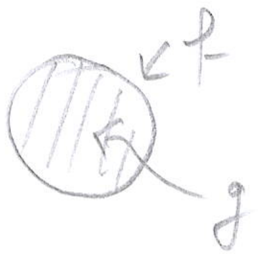
Stone-Weierstraß

Consider now $p \in B \subset A$, where $p(z) = z$

$\Rightarrow \sigma_A(p) = \mathbb{T} = \partial \mathbb{D}$

$\sigma_B(p) = \text{¿}$

note: $f \in B \xRightarrow[\text{analysis}]{\text{complex}}$ f can be extended to holomorphic (in particular, continuous) function g on $\mathbb{D}$, s.th. $g|\partial\mathbb{D} = f$



Consider now $\lambda \notin \sigma_B(p)$

$\Rightarrow \exists f \in B : (\lambda - z) f(z) = 1$ on $\mathbb{T}$

this extends to all of $\mathbb{D}$

$\Rightarrow \exists g \in C(\mathbb{D})$, $(g|\partial\mathbb{D} = f)$ s.th.

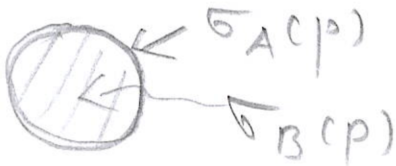
$(\lambda - z) g(z) = 1 \quad \forall z \in \mathbb{D}$

$\Rightarrow \lambda \neq z \quad \forall z \in \mathbb{D}$

$\Rightarrow \mathbb{D} \subset \sigma_B(p)$

Since $\sigma_B(p) \subset \{\lambda : |\lambda| \leq \underbrace{\|p\|}_{=1}\} = \mathbb{D}$

$\Rightarrow \sigma_B(p) = \mathbb{D}$ thus:



12.16. Theorem: Let A, B be Banach algebras with $B \subset A$ and consider $a \in B$. Then:

i) $\sigma_A(a) \subset \sigma_B(a)$

ii) $\partial\sigma_B(a) \subset \partial\sigma_A(a)$

Proof: i) $\lambda \notin \sigma_B(a) \Rightarrow \exists (\lambda - a)^{-1} \in B \subset A$

$\Rightarrow \lambda \notin \sigma_A(a)$

ii) Consider $\lambda \in \partial\sigma_B(a)$

to show: $\lambda \in \partial\sigma_A(a) = \sigma_A(a) \setminus \mathring{\sigma}_A(a)$

$$\left.\begin{array}{l} \text{Since } \mathring{\sigma}_A(a) \subset \mathring{\sigma}_B(a) \\ \lambda \notin \mathring{\sigma}_B(a) \end{array}\right\} \Rightarrow \lambda \notin \mathring{\sigma}_A(a)$$

remains to show: $\lambda \in \sigma_A(a)$

Assume $\lambda \notin \sigma_A(a)$, i.e. $\exists x \in A$ with

$x(\lambda - a) = (\lambda - a)x = 1$

$\lambda \in \partial\sigma_B(a) \Rightarrow \exists \lambda_n \in \mathbb{C} \setminus \sigma_B(a)$ with $\lambda_n \to \lambda$

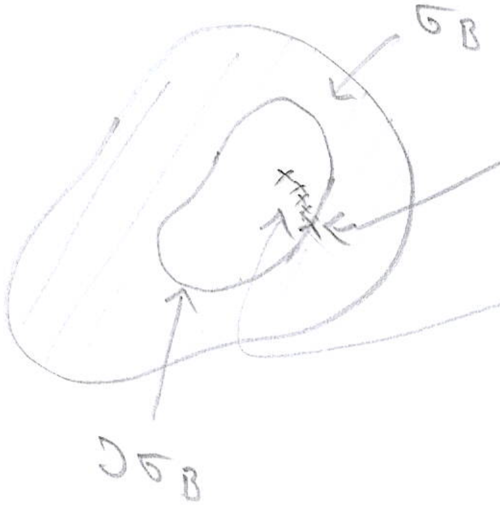
$(\lambda_n - a)^{-1} \in B \subset A$

$$\lambda_n \to \lambda \Rightarrow \lambda_n - a \to \lambda - a$$

$$\overset{\text{inversion}}{\underset{\text{continuous}}{\Longrightarrow}} \underbrace{(\lambda_n - a)^{-1}}_{\in B} \to (\lambda - a)^{-1} = x$$

$$B \text{ complete} \Rightarrow \underset{\substack{\| \\ (\lambda - a)^{-1}}}{x \in B} \Rightarrow \underset{\text{contradiction}}{\lambda \notin \sigma_B(a)}$$

$$\Rightarrow \lambda \in \sigma_A(a)$$



$\lambda \in \partial\sigma_B$, but $\lambda \notin \sigma_A$

$\Downarrow$

$\exists \lambda_n \in \rho_B$, i.e.

$$(\lambda_n - a)^{-1} \in B \xrightarrow{n \to \infty} (\lambda - a)^{-1} \in A$$

$\square$

12.17. Theorem: Let A be a commutative C^*-algebra and $a \in A$ selfadjoint, i.e. $a^* = a$. Then $\sigma(a) \subseteq \mathbb{R}$

Proof: later! (in 13.8)

12.18. Theorem: Let A and B be C^*-algebras with $B \subset A$ and $a \in B$. Then

$$\sigma_A(a) = \sigma_B(a)$$