**12.19. Remark:** For $a = a^* \in A$ put

$$B := C^*(a, 1) := \overline{\left\{ \alpha_0 \cdot 1 + \alpha_1 a + \dots + \alpha_n a^n \;\middle|\; \begin{matrix} n \in \mathbb{N} \\ \alpha_i \in \mathbb{C} \end{matrix} \right\}}$$

$$\subset A,$$

this is the $C^*$-algebra generated by $a$ (and $1$).

$\Rightarrow$ $B$ is commutative, thus

$$\sigma_A(a) \overset{12.18}{=} \sigma_B(a) \overset{12.17}{\subset} \mathbb{R}$$

i.e. 12.17 is also true for arbitrary $C^*$-algebra.

**Proof of 12.18:** Consider first $a = a^* \in B$, put

$C := C^*(a, 1)$ commutative $C^*$-algebra

and
$$C \subset B \subset A$$

We show: $\sigma_A(a) = \sigma_C(a) = \sigma_B(a)$

$$C \subset B \overset{12.16}{\Longrightarrow} \sigma_B(a) \subset \sigma_C(a) = \partial\sigma_C(a) \subset \partial\sigma_B(a) \subset \sigma_B(a)$$

since $\sigma_C(a) \subset \mathbb{R}$

$\Rightarrow$ $\sigma_B(a) = \sigma_C(a)$

in the same way: $\sigma_A(a) = \sigma_C(a)$

$\Bigg\}$ $\Rightarrow$ $\sigma_A(a) = \sigma_B(a)$

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Consider now arbitrary $a \in B$:

it suffices to show: $a$ invertible in $A$

$\Rightarrow a$ invertible in $B$

So let $a$ be invertible in $A$, i.e.

$\exists b \in A : ab = 1 = ba$

$\Rightarrow b^* a^* = 1 = a^* b^* \qquad (1^* = 1)$

$\Rightarrow (a^* a)(bb^*) = 1 = (bb^*)(a^* a)$

thus: $a^* a$ is invertible in $A$

$(a^* a)^* = a^* a$ is selfadjoint

$\overset{\text{above}}{\Longrightarrow} a^* a$ is invertible in $B$ and

$$bb^* = (a^* a)^{-1} \in B$$

$\Rightarrow b = b(b^* a^*) = \underbrace{(bb^*)}_{\in B} a^* \in B$

$\Rightarrow a^{-1} = b \in B$ $\qquad \square$