5. Banach spaces

5.1. Def: 1) A <u>normed space</u> is a vector space equipped with a norm.

2) A <u>Banach space</u> is a normed space that is complete with respect to its norm.

5.2. Examples: 1) $K \subset \mathbb{R}$ compact

$C(K) := \{ f : K \to \mathbb{F} \mid f \text{ continuous} \}$

is Banach space w.r.t

$$\|f\| := \max_{t \in K} |f(t)|$$

2) $1 \leq p \leq \infty$

$$L^p(a,b) := \{ f : [a,b] \to \mathbb{F} \mid \int_a^b |f(t)|^p \, dt < \infty \}$$

is Banach space w.r.t.

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}$$

$p = \infty$ is to be understood as follows:

$L^\infty(a,b) := \{ f : [a,b] \to \mathbb{F} \mid f \text{ (almost surely) bounded} \}$

$\|f\|_\infty :=$ essential supremum of $|f|$

L^p Hilbertspace $\iff$ $p = 2$!

3) discrete version of 2

$$\ell_p := \ell_p(\mathbb{N}) := \{ (\alpha_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |\alpha_n|^p < \infty \}$$

$$\|(\alpha_n)\|_p := \left(\sum |\alpha_n|^p\right)^{1/p}$$

$$\ell_\infty := \ell_\infty(\mathbb{N}) := \{ (\alpha_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |\alpha_n| < \infty \}$$

$$\|(\alpha_n)\|_\infty := \sup_{n\in\mathbb{N}} |\alpha_n|$$

ℓ_p is Banach space for $1 \le p \le \infty$

Hilbert space $p = 2$

4) closed subspaces of ℓ_∞:

$$c := \{ (\alpha_n) \mid \lim_{n\to\infty} \alpha_n \text{ exists} \}$$

$$c_0 := \{ (\alpha_n) \mid \lim_{n\to\infty} \alpha_n = 0 \}$$

5.3 Prop.: Let X be a normed space.

1) $+ : X \times X \to X$, $(x,y) \mapsto x+y$ is continuous

2) $\cdot : \mathbb{F} \times X \to X$, $(\alpha, x) \mapsto \alpha x$ is continuous

3) $\|\cdot\| : X \to \mathbb{F}$, $x \mapsto \|x\|$ is continuous

Proof: 1) $x_n \to x$, $y_n \to y$

$\|(x_n + y_n) - (x+y)\| \le \|x_n - x\| + \|y_n - y\|$

$\Rightarrow x_n + y_n \to x + y$

3) $|\, \|x\| - \|y\| \,| \le \|x - y\|$ (comp. 1.9) $\square$

5.4. Prop.: Let X and Y be normed spaces and $A : X \to Y$ a linear transformation. Then the following statements are equivalent:

a) A is continuous

b) A is continuous at 0.

c) There is a constant $c > 0$ s. th.

$$\|Ax\| \le c \|x\| \quad \forall x \in X$$

Proof: similar to 1.22, 2.1

b) $\Rightarrow$ c): A cont. at 0 $\Rightarrow \exists\, \delta > 0$ s.th.

$$\|x\| \le \delta \Rightarrow \|Ax\| \le 1$$

Put $c := \frac{1}{\delta}$ $\square$

5.5 Def.: Consider $A : X \to Y$ linear. We put

$$\|A\| := \sup_{\substack{x \in X \\ \|x\| = 1}} \|Ax\|$$ (operator) norm of A

$\|A\| < \infty \;:\Leftrightarrow$ A bounded

$B(X,Y) := \{ A : X \to Y \mid \|A\| < \infty \}$

In particular:

$B(X) := B(X,X)$

$X^* := B(X, \mathbb{F})$ <u>dual</u> (space) of X

<u>5.6. Theorem:</u> 1) Let X, Y be normed spaces

i) $B(X,Y)$ equipped with operator norm, pointwise addition and pointwise scalar multiplication is a normed space.

ii) If Y is a Banach space then $B(X,Y)$ is a Banach space, too.

In particular: X^* is Banach space for each normed space X.

2) Let X be a Banach space. Then $B(X)$ is a Banach algebra w.r.t. composition, i.e. we have

$\|AB\| \leq \|A\| \cdot \|B\|$

<u>Proof:</u> similar to 2.4.

in particular 1(ii): completeness

(A_n) CS in $B(X,Y)$ $\Rightarrow$ $(A_n x)$ CS in Y $\forall x \in X$

$\overset{Y\ \text{BS}}{\Longrightarrow}$ $\lim\limits_{n\to\infty} A_n x =: Ax$ exists

show: $A \in B(X,Y)$, $A_n \to A$ as in 2.4. $\square$

5.7. Example for dual spaces: We have

$$c_0^* = c^* = \ell_1, \quad \ell_1^* = \ell_\infty$$

$$\ell_p^* = \ell_q \quad \text{for } 1 < p < \infty$$

$$\text{where } \frac{1}{p} + \frac{1}{q} = 1$$

(i.e. $p = 2 \Rightarrow q = 2 \Rightarrow \ell_2^* = \ell_2$)

diffrent phenomena:

$$c_0 \neq c, \quad \text{but} \quad c_0^* = c^*$$

$1 < p < \infty$: $\ell_p^{**} = (\ell_p^*)^* = \ell_q^* = \ell_p$, i.e. $\ell_p^{**} = \ell_p$

Banach spaces are called reflexive, if $X^{**} = X$

thus: ℓ_p is, for $1 < p < \infty$, reflexive

$c_0, c, \ell_1, \ell_\infty$ are not reflexive

(e.g.: $c^{**} = \ell_1^* = \ell_\infty \neq c$)

Proof of $c_0^* = \ell_1$:

Consider $x \in \ell_1$, i.e. $x = (\beta_n)$ s.th. $\sum |\beta_n| < \infty$

Define $L_x : c_0 \to \mathbb{F}$

$$(\alpha_n) \mapsto \sum_{n \in \mathbb{N}} \alpha_n \beta_n \qquad y = (\alpha_n) \in c_0$$

$$\Rightarrow |L_x(y)| = |\sum \alpha_n \beta_n| \leq \sum |\alpha_n| \, |\beta_n|$$

$$\leq \|y\| \underbrace{\sum |\beta_n|}_{\|x\|_1} \qquad \|y\| = \sup_{n \in \mathbb{N}} |\alpha_n|$$

$\Rightarrow \quad \|L_x\| \leq \|x\|_1$, i.e. $L_x \in c_0^*$

The other way round: Consider $L \in c_0^*$

Denote $e_n := (0, \dots, 0, 1, 0, \dots) \in c_0$
$\uparrow$ n-th position

and put

$$\beta_n := L(e_n)$$

Claim: $L((\alpha_n)) = \sum \alpha_n \beta_n$, i.e. $L = L_x$, where $x = (\beta_n)$

Consider $y = (\alpha_n)_{n \in \mathbb{N}} \in c_0$

Put $y_n := (\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, 0, \dots)$

$$= \sum_{k=1}^{n} \alpha_k e_k$$

We have: $y_n \to y$ in c_0, because

$$\|y_n - y\|_\infty = \left\| \sum_{k=n+1}^{\infty} \alpha_k e_k \right\|_\infty = \|(0, \dots, 0, \alpha_{n+1}, \alpha_{n+2}, \dots)\|_\infty$$

$$= \sup_{k > n} |\alpha_k|$$

$$\xrightarrow[n \to \infty]{} 0$$

(since $y \in c_0$, i.e. $\lim_{n \to \infty} \alpha_n = 0$)

thus: $y_n \to y \overset{L \text{ cont.}}{\Longrightarrow} L(y_n) \xrightarrow{n \to \infty} L(y)$

$$L(y_n) = L\left(\sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k L(e_k) = \sum_{k=1}^{n} \alpha_k \beta_k$$

$$\Rightarrow L(y) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k \beta_k = \sum_{k=1}^{\infty} \alpha_k \beta_k$$

thus: $L \in c_0^* \rightarrow \exists\, x$ with $L = L_x$

remains to show: $x \in \ell_1$

assume: $x \notin \ell_1$, i.e. $\sum |\beta_n| = \infty$

But we know: $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty \quad \forall\, (\alpha_n) \in c_0$

$$(\alpha_n) \rightsquigarrow \left(\alpha_n \frac{|\beta_n|}{\beta_n}\right) \Longrightarrow \sum_{n=1}^{\infty} \alpha_n \cdot |\beta_n| < \infty \quad \forall\, (\alpha_n) \in c_0$$

Choose N_n s.th

$$n < \sum_{k=N_n+1}^{N_{n+1}} |\beta_k| \leq n+1 \qquad (N_1 = 0)$$

and put

$$\alpha_k = \frac{1}{n} \quad \text{for } N_n + 1 \leq k \leq N_{n+1} \qquad (\Rightarrow (\alpha_k) \in c_0)$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n |\beta_n| = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=N_n+1}^{N_{n+1}} |\beta_k| \geq \sum_{n=1}^{\infty} 1 = \infty$$

contradiction !

$\Rightarrow x \in \ell_1$ %

Thus we have isomorphism

$$\ell_1 \longrightarrow c_0^*$$
$$x \longmapsto L_x$$

$$L_x(y) = \sum \alpha_n \beta_n = \langle \underset{\substack{\| \\ (\alpha_n) \in c_0}}{y}, \overset{(\beta_n) \in \ell_1}{x} \rangle$$

we even have

$$\|x\|_1 = \|L_x\|$$

because: $\|L_x\| \le \|x\|_1$ by above

and $\|L_x\| \geq |L_x(y)|$ if $y \in c_0$ with $\|y\|_\infty = 1$

Choose $y = \sum_{k=1}^{n} e_k \frac{|\beta_k|}{\beta_k}$

$$\Rightarrow L_x(y) = \sum_{k=1}^{n} \underbrace{L_x(e_k)}_{\beta_k} \frac{|\beta_k|}{\beta_k} = \sum_{k=1}^{n} |\beta_k| \xrightarrow{n \to \infty} \|x\|_1$$

$$\Rightarrow \|L_x\| \geq \|x\|_1$$

thus: $\|L_x\| = \|x\|_1$

we have thus shown that c_0^* is isometrically isomorphic to ℓ_1

<u>5.8 Def.</u>: Let X and Y be normed spaces.

1) X and Y are <u>isometrically isomorphic</u> if there exists a linear isometric bijection

$$T: X \to Y$$

(i.e: $\|Tx\|_Y = \|x\|_X$)

2) X and Y are <u>isomorphic</u> if there exists a linear bijection $T: X \to Y$ which is a homeomorphism (i.e. T, T^{-1} are continuous)

<u>5.9. Remark</u>: Isometric isomorphism preserves norms, isomorphism maps norm in equivalent norm $\left(\frac{1}{\|T^{-1}\|} \|x\| \leq \|Tx\| \leq \|T\| \, \|x\| \right)$, i.e. it preserves the topology.