5. Banach spaces

5.1. Def: 1) A normed space is a vector space equipped with a norm.

2) A Banach space is a normed space that is complete with respect to its norm.

5.2. Examples: 1) $K \subset \mathbb{R}$ compact

$C(K) := \{ f : K \to \mathbb{F} \mid f \text{ continuous} \}$

is Banach space w.r.t.

$$\|f\| := \max_{t \in K} |f(t)|$$

2) $1 \leq p \leq \infty$

$$L^p(a,b) := \{ f : [a,b] \to \mathbb{F} \mid \int_a^b |f(t)|^p \, dt < \infty \}$$

is Banach space w.r.t.

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}$$

$p = \infty$ is to be understood as follows:

$$L^\infty(a,b) := \{ f : [a,b] \to \mathbb{F} \mid f \text{ (almost surely) bounded} \}$$

$\|f\|_\infty :=$ essential supremum of $|f|$

L^p Hilbertspace $\Leftrightarrow$ $p = 2$!

3) discrete version of 2

$$\ell_p := \ell_p(\mathbb{N}) := \{ (\alpha_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |\alpha_n|^p < \infty \}$$

$$\|(\alpha_n)\|_p := \left(\sum |\alpha_n|^p\right)^{1/p}$$

$$\ell_\infty := \ell_\infty(\mathbb{N}) := \{ (\alpha_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |\alpha_n| < \infty \}$$

$$\|(\alpha_n)\|_\infty := \sup_{n\in\mathbb{N}} |\alpha_n|$$

ℓ_p is Banach space for $1 \leq p \leq \infty$

Hilbert space $p = 2$

4) closed subspaces of ℓ_∞:

$$c := \{ (\alpha_n) \mid \lim_{n\to\infty} \alpha_n \text{ exists} \}$$

$$c_0 := \{ (\alpha_n) \mid \lim_{n\to\infty} \alpha_n = 0 \}$$

5.3 Prop.: Let X be a normed space.

1) $+ : X \times X \to X$, $(x,y) \mapsto x+y$ is continuous

2) $\cdot : \mathbb{F} \times X \to X$, $(\alpha, x) \mapsto \alpha x$ is continuous

3) $\|\cdot\| : X \to \mathbb{F}$, $x \mapsto \|x\|$ is continuous

Proof: 1) $x_n \to x$, $y_n \to y$

$\| (x_n + y_n) - (x+y) \| \leq \| x_n - x \| + \| y_n - y \|$

$\Rightarrow x_n + y_n \to x+y$

3) $| \|x\| - \|y\| | \leq \| x - y \|$ (comp. 1.9) $\square$

5.4 Prop.: Let X and Y be normed spaces and $A: X \to Y$ a linear transformation. Then the following statements are equivalent:

a) A is continuous

b) A is continuous at 0.

c) There is a constant $c > 0$ s. th.

$$\| Ax \| \leq c \| x \| \quad \forall x \in X$$

Proof: similar to 1.22, 2.1

b) $\Rightarrow$ c): A cont. at 0 $\Rightarrow \exists\, \delta > 0$ s.th.

$$\|x\| \leq \delta \Rightarrow \|Ax\| \leq 1$$

Put $c := \frac{1}{\delta}$ $\square$

5.5 Def.: Consider $A: X \to Y$ linear. We put

$\|A\| := \sup\limits_{\substack{x \in X \\ \|x\|=1}} \|Ax\|$ (operator) norm of A

$\|A\| < \infty \;:\Leftrightarrow$ A bounded

$B(X,Y) := \{ A : X \to Y \mid \|A\| < \infty \}$

In particular:

$B(X) := B(X,X)$

$X^* := B(X, \mathbb{F})$ <u>dual</u> (space) of X

<u>5.6. Theorem:</u> 1) Let X, Y be normed spaces

i) $B(X,Y)$ equipped with operator norm, pointwise addition and pointwise scalar multiplication is a normed space.

ii) If Y is a Banach space then $B(X,Y)$ is a Banach space, too.

In particular: X^* is Banach space for each normed space X.

2) Let X be a Banach space. Then $B(X)$ is a Banach algebra w.r.t. composition, i.e. we have

$\| AB\| \leq \|A\| \cdot \|B\|$

<u>Proof</u>: similar to 2.4.

in particular 1(ii): completeness

(A_n) CS in $B(X,Y)$ $\Rightarrow$ $(A_n x)$ CS in Y $\forall x \in X$

$\overset{Y \text{ BS}}{\Longrightarrow}$ $\lim\limits_{n\to\infty} A_n x =: Ax$ exists

show: $A \in B(X,Y)$, $A_n \to A$ as in 2.4. □