5.7. Example for dual spaces: We have

$$c_0^* = c^* = \ell_1, \quad \ell_1^* = \ell_\infty$$

$$\ell_p^* = \ell_q \quad \text{for } 1 < p < \infty$$

$$\text{where } \frac{1}{p} + \frac{1}{q} = 1$$

(i.e. $p = 2 \Rightarrow q = 2 \Rightarrow \ell_2^* = \ell_2$)

diffrent phenomena:

$c_0 \neq c$, but $c_0^* = c^*$

$1 < p < \infty$: $\ell_p^{**} = (\ell_p^*)^* = \ell_q^* = \ell_p$, i.e. $\ell_p^{**} = \ell_p$

Banach spaces are called reflexive, if $X^{**} = X$

thus: ℓ_p is, for $1 < p < \infty$, reflexive

$c_0, c, \ell_1, \ell_\infty$ are not reflexive

(e.g.: $c^{**} = \ell_1^* = \ell_\infty \neq c$)

Proof of $c_0^* = \ell_1$:

Consider $x \in \ell_1$, i.e. $x = (\beta_n)$ s.th. $\sum |\beta_n| < \infty$

Define $L_x : c_0 \to \mathbb{F}$

$$(\alpha_n) \mapsto \sum_{n \in \mathbb{N}} \alpha_n \beta_n \qquad y = (\alpha_n) \in c_0$$

$$\Rightarrow |L_x(y)| = |\sum \alpha_n \beta_n| \leq \sum |\alpha_n| |\beta_n|$$

$$\leq \|y\| \underbrace{\sum |\beta_n|}_{\|x\|_1} \qquad \|y\| = \sup_{n \in \mathbb{N}} |\alpha_n|$$

$\Rightarrow \|L_x\| \leq \|x\|_1$, i.e. $L_x \in c_0^*$

The other way round: Consider $L \in c_0^*$

Denote $e_n := (0, \dots, 0, \underset{\substack{\uparrow \\ \text{n-th position}}}{1}, 0, \dots) \in c_0$

and put

$\beta_n := L(e_n)$

Claim: $L((\alpha_n)) = \sum \alpha_n \beta_n$, i.e. $L = L_x$, where $x = (\beta_n)$

Consider $y = (\alpha_n)_{n \in \mathbb{N}} \in c_0$

Put $y_n := (\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, 0, \dots)$

$$= \sum_{k=1}^{n} \alpha_k e_k$$

We have: $y_n \to y$ in c_0, because

$$\|y_n - y\|_\infty = \|\sum_{k=n+1}^{\infty} \alpha_k e_k\|_\infty = \|(0, \dots, 0, \alpha_{n+1}, \alpha_{n+2}, \dots)\|_\infty$$
$$= \sup_{k > n} |\alpha_k|$$

$$\xrightarrow[n \to \infty]{} 0$$

(since $y \in c_0$, i.e. $\lim_{n \to \infty} \alpha_n = 0$)

thus: $y_n \to y \overset{L \text{ cont.}}{\Longrightarrow} L(y_n) \xrightarrow{n \to \infty} L(y)$

$$\overset{\|}{L(\sum_{k=1}^{n} \alpha_k e_k)} = \sum_{k=1}^{n} \alpha_k L(e_k) = \sum_{k=1}^{n} \alpha_k \beta_k$$

$$\Rightarrow L(y) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k \beta_k = \sum_{k=1}^{\infty} \alpha_k \beta_k$$

thus: $L \in c_0^* \to \exists\, x$ with $L = L_x$

remains to show: $x \in \ell_1$

assume: $x \notin \ell_1$, i.e. $\sum |\beta_n| = \infty$

But we know: $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty \quad \forall\, (\alpha_n) \in c_0$

$$(\alpha_n) \rightsquigarrow \left(\alpha_n \frac{|\beta_n|}{\beta_n}\right) \Longrightarrow \sum_{n=1}^{\infty} \alpha_n \cdot |\beta_n| < \infty \quad \forall\, (\alpha_n) \in c_0$$

Choose N_n s.th

$$n < \sum_{k=N_n+1}^{N_{n+1}} |\beta_k| \leq n+1 \qquad (N_1 = 0)$$

and put

$$\alpha_k = \frac{1}{n} \quad \text{for } N_n + 1 \leq k \leq N_{n+1} \qquad (\Rightarrow (\alpha_k) \in c_0)$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n |\beta_n| = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=N_n+1}^{N_{n+1}} |\beta_k| \geq \sum_{n=1}^{\infty} 1 = \infty$$

contradiction ↯

$\Rightarrow x \in \ell_1$ %

Thus we have isomorphism

$$\ell_1 \longrightarrow c_0^*$$
$$x \longmapsto L_x$$

$$L_x(y) = \sum \alpha_n \beta_n = \langle \underbrace{y}_{(\alpha_n) \in c_0}, \overbrace{x}^{(\beta_n) \in \ell_1} \rangle$$

we even have

$$\|x\|_1 = \|L_x\|$$

because: $\|L_x\| \leq \|x\|_1$ by above

and $\|L_x\| \geq |L_x(y)|$ if $y \in c_0$ with $\|y\|_\infty = 1$

Choose $y = \sum_{k=1}^{n} e_k \frac{|\beta_k|}{\beta_k}$

$$\Rightarrow L_x(y) = \sum_{k=1}^{n} \underbrace{L_x(e_k)}_{\beta_k} \frac{|\beta_k|}{\beta_k} = \sum_{k=1}^{n} |\beta_k| \xrightarrow{n \to \infty} \|x\|_1$$

$$\Rightarrow \|L_x\| \geq \|x\|_1$$

thus: $\|L_x\| = \|x\|_1$

we have thus shown that c_0^* is isometrically isomorphic to ℓ_1

<u>5.8 Def.</u>: Let X and Y be normed spaces.

1) X and Y are <u>isometrically isomorphic</u> if there exists a linear isometric bijection

$$T : X \to Y$$

(i.e: $\|Tx\|_Y = \|x\|_X$)

2) X and Y are <u>isomorphic</u> if there exists a linear bijection $T: X \to Y$ which is a homeomorphism (i.e. T, T^{-1} are continuous)

<u>5.9. Remark</u>: Isometric isomorphism preserves norms, isomorphism maps norm in equivalent norm ($\frac{1}{\|T^{-1}\|} \|x\| \leq \|Tx\| \leq \|T\| \, \|x\|$), i.e. it preserves the topology.