2. Operators on Hilbert spaces

operator $\hat{=}$ linear transformation $\hat{=}$ linear function

2.1. Prop.: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $A: \mathcal{H} \to \mathcal{K}$ a linear transformation. The following statements are equivalent:

(a) A is continuous.

(b) A is continuous at 0.

(c) There is a constant $c \geq 0$ s.th

$$\|Ax\| \leq c\,\|x\| \qquad \forall x \in \mathcal{H}$$

Proof: similar to 1.22.

(b) $\Rightarrow$ (c): A continuous at 0, i.e. $\exists \delta > 0$ s.th

$$\|y\| \leq \delta \Rightarrow \|Ay\| \leq 1$$

Put $c := \frac{1}{\delta}$

(c) $\Rightarrow$ (a): Consider $x_n \to x$ (i.e. $\|x - x_n\| \to 0$)

$$\Rightarrow \|A(x-x_n)\| \leq c\,\|x-x_n\| \to 0$$

$$\|A(x-x_n)\| = \|Ax - Ax_n\|$$

$$\Rightarrow Ax_n \to Ax$$

<u>2.2. Def.:</u> Let $A : \mathcal{H} \to \mathcal{K}$ be a linear operator.

We put

$$\|A\| := \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

and call it (operator) <u>norm</u> of A.

A is called <u>bounded</u> if $\|A\| < \infty$

$B(\mathcal{H}, \mathcal{K}) := \{ A : \mathcal{H} \to \mathcal{K} \text{ linear} \mid A \text{ bounded} \}$

$B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$ <u>bounded operators</u>

<u>2.3. Remark:</u> $\|A\|$ is the best (smallest possible) constant c in $\|Ax\| \leq c\|x\|$, in particular

$$\|Ax\| \leq \|A\| \cdot \|x\|.$$

We will consider in the following $B(\mathcal{H})$, most things are also true for $B(\mathcal{H}, \mathcal{K})$!

<u>2.4. Theorem:</u> 1) $B(\mathcal{H})$ is vector space over $\mathbb{F}$ w.r.t.

$$(A+B)(x) := Ax + Bx \qquad (A, B \in B(\mathcal{H}), \lambda \in \mathbb{F})$$
$$(\lambda A)(x) := \lambda Ax$$

2) $\|\cdot\|$ is a norm on $B(\mathcal{H})$

3) $B(\mathcal{H})$ is complete, i.e. Banach space

4) $B(\mathcal{H})$ is algebra w.r.t. composition

$$(AB)(x) := A(Bx) \qquad (A, B \in B(\mathcal{H}))$$

and we have

$$\|AB\| \leq \|A\| \cdot \|B\| \qquad \forall A, B \in B(\mathcal{H})$$

($B(\mathcal{H})$ is "Banach algebra")

<u>Proof:</u> 1) direct checking

2) direct checking, e.g.

$$\|A+B\| = \sup_{\|x\|=1} \underbrace{\|(A+B)x\|}_{\|Ax+Bx\| \leq \|Ax\| + \|Bx\|}$$

$$\leq \sup_{\|x\|=1} \|Ax\| + \sup_{\|x\|=1} \|Bx\|$$

$$= \|A\| + \|B\|$$

$$\|A\| = 0 \Rightarrow \|Ax\| \leq 0 \cdot \|x\| = 0$$

$$\Rightarrow \|Ax\| = 0$$

$$\Rightarrow Ax = 0 \qquad \forall x \in \mathcal{H}$$

i.e. $A = 0$

3) Let $(A_n)_{n\in\mathbb{N}}$ be Cauchy sequence

to show: $\exists A \in B(\mathcal{H})$ s.th. $A_n \to A$

Consider $x \in \mathcal{H}$:

$$\|A_n x - A_m x\| = \|(A_n - A_m)x\| \leq \|A_n - A_m\| \cdot \|x\|$$

$\Rightarrow (A_n x)_{n \in \mathbb{N}}$ CS

$\mathcal{H}$ complete $\Rightarrow \exists y \in \mathcal{H}$ s.th. $A_n x \to y$

Define $A : \mathcal{H} \to \mathcal{H}$ by $Ax := y$

A is linear : $\left.\begin{array}{l} A_n x_1 \to y_1 \\ A_n x_2 \to y_2 \end{array}\right\} \Rightarrow A_n(x_1 + x_2) \to y_1 + y_2$

Furthermore:

$$\|Ax - A_m x\| \leq \underbrace{\|Ax - A_n x\|}_{\substack{\text{arbitrarily small} \\ \text{for } n \text{ suff. large}}} + \underbrace{\|A_n x - A_m x\|}_{\leq \underbrace{\|A_n - A_m\|}_{\leq \frac{\varepsilon}{2} \text{ if } n,m \geq N(\varepsilon)} \cdot \|x\|}$$

$\overset{n\to\infty}{\Rightarrow} \|Ax - A_m x\| \leq \varepsilon \|x\|$ if $m \geq N(\varepsilon)$ (indep. of x)

$$\Rightarrow \|Ax\| \leq \underbrace{\|A_m x\|}_{\leq \|A_m\| \cdot \|x\|} + \underbrace{\|Ax - A_m x\|}_{\leq \varepsilon \|x\|}$$

$\Rightarrow \|A\| \leq \|A_m\| + \varepsilon$, i.e. $A \in B(\mathcal{H})$

and $\|(A - A_m)x\| \leq \varepsilon \|x\|$

i.e. $\|A - A_m\| \leq \varepsilon$ if $m \geq N(\varepsilon)$

$\Rightarrow A_m \to A$

4) $\|ABx\| \leq \|A\| \, \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\|$

$\Rightarrow \|AB\| \leq \|A\| \cdot \|B\|$

$\Rightarrow A \cdot B \in B(\mathcal{H})$ $\square$

2.5. Examples: 1) $\mathcal{H} = \mathbb{F}^n$, $(e_i)_{i=1}^n$ ONB of $\mathcal{H}$

$$A \in B(\mathcal{H}) \Rightarrow Ae_k = \sum_{l=1}^n \alpha_{kl} e_l$$

$$\Rightarrow A \mathrel{\hat{=}} (\alpha_{kl})_{k,l=1}^n$$

$B(\mathcal{H}) \mathrel{\hat{=}} M_{n \times n}(\mathbb{F})$ $n \times n$ - matrices

2) $\mathcal{H} = L^2(a,b) = \{ f : [a,b] \to \mathbb{F} \mid \int_a^b |f(t)|^2 dt < \infty \}$

Consider

$k \in L^2([a,b] \times [a,b])$, i.e. $\int_a^b \int_a^b |k(s,t)|^2 \, ds \, dt < \infty$

and define

$K : L^2(a,b) \to L^2(a,b)$ by

$$(Kf)(s) := \int_a^b k(s,t) f(t) \, dt$$

Then: $K \in B(L^2(a,b))$ and $\|K\| \leq \|k\|_{L^2([a,b] \times [a,b])}$

because:

$$\|Kf\|^2 = \int |(Kf)(s)|^2 ds$$

$$= \int \underbrace{\left| \int k(s,t) f(t) \, dt \right|^2}_{\overset{C-S}{\leq} \int |k(s,t)|^2 dt \cdot \int |f(t)|^2 dt} ds$$

$$\Rightarrow \|kf\|^2 \le \underbrace{\iint |k(s,t)|^2 \, dt\, ds}_{\|k\|^2_{L^2([a,b]\times[a,b])}} \cdot \underbrace{\int |f(t)|^2 \, dt}_{\|f\|^2_{L^2[a,b]}}$$

k is called <u>integral operator</u> (or Hilbert-Schmidt operator) with kernel k.

note: kernels $\hat{=}$ continuous analogues of matrices

but: not each bounded operator is integral operator

e.g. $\dim \mathcal{H} = \infty \Rightarrow id$ is not an integral operator

3) special integral operator

$\mathcal{H} = L^2(0,1)$

$$k(s,t) = \begin{cases} 0 & s \le t \\ 1 & s > t \end{cases}$$

$$(Vf)(s) = \int_0^s f(t)\, dt \qquad (0 \le s \le 1)$$

V is called Volterra-operator

$\|V\| = \frac{2}{\pi}$ non trivial

(by (2), we know $\|V\| \le \frac{1}{\sqrt{2}}$)

2.6. Theorem: Let $A \in B(\mathcal{H})$. There exists a unique operator $A^* \in B(\mathcal{H})$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \forall x, y \in \mathcal{H}$$

2.7. Def.: A^* is called the adjoint of A.

Proof: Fix $y \in \mathcal{H}$; consider

$$L: \mathcal{H} \to \mathbb{F}$$
$$x \mapsto \langle Ax, y\rangle$$

then: - L linear

- L bounded: $|Lx| = |\langle Ax, y\rangle|$

$$\leq \|Ax\| \cdot \|y\|$$
$$\leq \|A\| \, \|x\| \, \|y\|$$

$$\Rightarrow L \in \mathcal{H}^*, \text{ and } \|L\| \leq \|A\| \, \|y\|$$

$$\overset{\text{Riesz}}{\underset{1.25}{\Rightarrow}} \exists!\, y_0 \in \mathcal{H} \text{ s.th. } \underset{\langle Ax, y\rangle}{\underset{\|}{L(x)}} = \langle x, y_0\rangle$$

Define $A^*: \mathcal{H} \to \mathcal{H}$ by $A^* y = y_0$

then: - A^* linear

- A^* bounded: $\underset{\|A^* y\|}{\underset{\|}{\|y_0\|}} \overset{1.25}{=} \|L\| \leq \|A\| \, \|y\|$

$$\Rightarrow \|A^*\| \leq \|A\| \Rightarrow A^* \in B(\mathcal{H}) \quad \square$$

2.8. Prop.: If $A, B \in B(\mathcal{H})$ and $\alpha, \beta \in \mathbb{F}$, then

1) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$

2) $(AB)^* = B^* A^*$

3) $A^{**} = A$

4) If A is invertible in $B(\mathcal{H})$
(i.e. $\exists A^{-1} \in B(\mathcal{H})$ s.th. $A A^{-1} = A^{-1} A = id$),
then A^* is invertible and
$(A^*)^{-1} = (A^{-1})^*$

5) $\|A\| = \|A^*\|$

6) $\|A\|^2 = \|A A^*\|$

2.9. Remarks: 1) Banach algebra + (1), .., (5)
= Banach $*$-algebra

Banach $*$-algebra + (6) = C^*-algebra

2) $\mathcal{H} = \mathbb{C}^n$, then $B(\mathbb{C}^n) \cong M_n(\mathbb{C})$ and

$A = (a_{ij})_{i,j=1}^n \Rightarrow A^* = (\bar{a}_{ji})_{i,j=1}^n$

3) $\mathcal{H} = L^2(a,b)$

K integral operator with kernel k

$\Rightarrow K^*$ —— " —— k^*

where $k^*(s,t) = \overline{k(t,s)}$

<u>Proof</u>: 1) – 4) direct checking!

e.g.: $\langle (\alpha A + \beta B)x, y\rangle = \langle x, (\alpha A + \beta B)^* y\rangle$

$$\|$$

$\alpha \langle Ax, y\rangle + \beta \langle Bx, y\rangle = \alpha \langle x, A^* y\rangle + \beta \langle x, B^* y\rangle$

$= \langle x, (\bar{\alpha} A^* + \bar{\beta} B^*) y\rangle$

$$\left.\begin{array}{l} \text{5) Proof of 2.6.} \Rightarrow \|A^*\| \leq \|A\| \\ A \rightsquigarrow A^* \Rightarrow \|A\| \underset{3)}{=} \|A^{**}\| \leq \|A^*\| \end{array}\right\} \Rightarrow \|A\| = \|A^*\|$$

6) $\|Ax\|^2 = \langle Ax, Ax\rangle$

$= \langle A^* A x, x\rangle$

$\overset{CS}{\leq} \|A^* A x\| \, \|x\|$

$\leq \|A^* A\| \, \|x\|^2$

$\Longrightarrow \|A\|^2 \leq \|A^* A\| \leq \|A^*\| \, \|A\| \overset{5)}{=} \|A\|^2$

$\Rightarrow$ every where "="

$\Rightarrow \|A^* A\| = \|A\|^2$ $\square$

<u>2.10. Prop.</u>: If $A \in B(\mathcal{H})$, then

$\ker A = (\operatorname{ran} A^*)^\perp$

<u>2.11. Remark</u>: note: $\ker A$, $A^\perp$ are always closed, but not $\operatorname{ran} A$ in general

thus: $\ker A^\perp = (\operatorname{ran} A^*)^{\perp\perp} = \overline{\operatorname{ran} A^*}$

<u>Proof</u>: "$\subset$" : Consider $x \in \ker A$

to show: $x \in (\operatorname{ran} A^*)^\perp$, i.e. $x \perp y \quad \forall y = A^* z \in \operatorname{ran} A^*$

$$\langle x, y \rangle = \langle x, A^* z \rangle = \langle \underbrace{Ax}_{=0}, z \rangle = 0$$

$\Rightarrow \ker A \subset (\operatorname{ran} A^*)^\perp$

"$\supset$": Consider $x \in (\operatorname{ran} A^*)^\perp$, i.e.

$x \perp y \quad \forall y = A^* z \in \operatorname{ran} A^*$

i.e. $\langle x, A^* z \rangle = 0 \quad \forall z \in \mathcal{H}$

$$\begin{array}{c} \langle x, A^* z \rangle \\ \| \\ \langle Ax, z \rangle \end{array} \Rightarrow Ax = 0$$

$\Rightarrow x \in \ker A$

$\Rightarrow (\operatorname{ran} A^*)^\perp \subset \ker A$ $\square$

<u>2.12. Def.</u>: 1) $A \in B(\mathcal{H})$ is <u>self-adjoint</u> (or <u>hermitian</u>) if $A = A^*$

2) $A \in B(\mathcal{H})$ is <u>normal</u>, if $AA^* = A^*A$

3) $\mathcal{U} \in B(\mathcal{H})$ is <u>unitary</u> if $\mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = \mathrm{id}$

4) $V \in B(\mathcal{H})$ is <u>isometric</u> if $V^*V = \mathrm{id}$

5) $P \in B(\mathcal{H})$ is a (orthogonal) <u>projection</u> if $P^* = P = P^2$

2.13. Remark: isometric means

$$\|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, V^*Vx\rangle = \langle x, x\rangle = \|x\|^2$$

if $\dim \mathcal{H} = \infty$: isometric $\not\Longrightarrow$ surjective

(compare one-sided shift 1.44)

isometric + surjective $\Longleftrightarrow$ unitary

2.14 Prop.: 1) Consider $A = A^* \in B(\mathcal{H})$.

i) $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$

ii) $\langle Ax, x\rangle = 0 \quad \forall x \in \mathcal{H} \Rightarrow A = 0$

2) If $\mathbb{F} = \mathbb{C}$ and $A \in B(\mathcal{H})$, then:

i) $\langle Ax, x\rangle = 0 \quad \forall x \in \mathcal{H} \Rightarrow A = 0$

ii) $A = A^* \Longleftrightarrow \langle Ax, x\rangle \in \mathbb{R} \quad \forall x \in \mathcal{H}$

Proof: Exercise!