2. Operators on Hilbert spaces

operator $\hat{=}$ linear transformation $\hat{=}$ linear function

2.1. Prop.: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $A: \mathcal{H} \to \mathcal{K}$ a linear transformation. The following statements are equivalent:

(a) A is continuous.

(b) A is continuous at 0.

(c) There is a constant $c \geq 0$ s.th.

$$\|Ax\| \leq c\,\|x\| \qquad \forall x \in \mathcal{H}$$

Proof: similar to 1.22.

(b) $\Rightarrow$ (c): A continuous at 0, i.e. $\exists \delta > 0$ s.th.

$$\|y\| \leq \delta \Rightarrow \|Ay\| \leq 1$$

Put $c := \frac{1}{\delta}$

(c) $\Rightarrow$ (a): Consider $x_n \to x$ (i.e. $\|x - x_n\| \to 0$)

$$\Rightarrow \underset{\|Ax - Ax_n\|}{\underset{\parallel}{\|A(x-x_n)\|}} \leq c\,\|x - x_n\| \to 0$$

$$\Rightarrow Ax_n \to Ax$$

□

2.2. Def.: Let $A : \mathcal{H} \to \mathcal{K}$ be a linear operator.

We put

$$\|A\| := \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

and call it (operator) norm of A.

A is called bounded if $\|A\| < \infty$

$B(\mathcal{H}, \mathcal{K}) := \{ A : \mathcal{H} \to \mathcal{K} \text{ linear} \mid A \text{ bounded} \}$

$B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$ bounded operators

2.3. Remark: $\|A\|$ is the best (smallest possible) constant c in $\|Ax\| \leq c\|x\|$, in particular

$$\|Ax\| \leq \|A\| \cdot \|x\|.$$

We will consider in the following $B(\mathcal{H})$, most things are also true for $B(\mathcal{H}, \mathcal{K})$!

2.4. Theorem: 1) $B(\mathcal{H})$ is vector space over $\mathbb{F}$ w.r.t.

$$(A+B)(x) := Ax + Bx$$
$$(\lambda A)(x) := \lambda \cdot Ax \qquad (A, B \in B(\mathcal{H}), \lambda \in \mathbb{F})$$

2) $\|\cdot\|$ is a norm on $B(\mathcal{H})$

3) $B(\mathcal{H})$ is complete, i.e. Banach space

4) $B(\mathcal{H})$ is algebra w.r.t. composition

$$(AB)(x) := A(Bx) \qquad (A, B \in B(\mathcal{H}))$$

and we have

$$\|AB\| \le \|A\| \cdot \|B\| \qquad \forall A, B \in B(\mathcal{H})$$

($B(\mathcal{H})$ is "Banach algebra")

<u>Proof:</u> 1) direct checking

2) direct checking, e.g.

$$\|A+B\| = \sup_{\|x\|=1} \underbrace{\|(A+B)x\|}_{\|Ax + Bx\| \le \|Ax\| + \|Bx\|}$$

$$\le \sup_{\|x\|=1} \|Ax\| + \sup_{\|x\|=1} \|Bx\|$$

$$= \|A\| + \|B\|$$

$$\|A\| = 0 \Rightarrow \|Ax\| \le 0 \cdot \|x\| = 0$$

$$\Rightarrow \|Ax\| = 0$$

$$\Rightarrow Ax = 0 \quad \forall x \in \mathcal{H}$$

i.e. $A = 0$

3) Let $(A_n)_{n \in \mathbb{N}}$ be Cauchy sequence

to show: $\exists A \in B(\mathcal{H})$ s.th. $A_n \to A$

Consider $x \in \mathcal{H}$:

$$\|A_n x - A_m x\| = \|(A_n - A_m)x\| \leq \|A_n - A_m\| \cdot \|x\|$$

$\Rightarrow (A_n x)_{n \in \mathbb{N}}$ CS

$\mathcal{H}$ complete $\Rightarrow \exists\, y \in \mathcal{H}$ s.th. $A_n x \to y$

Define $A : \mathcal{H} \to \mathcal{H}$ by $Ax := y$

A is linear : $\left.\begin{array}{l} A_n x_1 \to y_1 \\ A_n x_2 \to y_2 \end{array}\right\} \Rightarrow A_n(x_1 + x_2) \to y_1 + y_2$

Furthermore:

$$\|Ax - A_m x\| \leq \underbrace{\|Ax - A_n x\|}_{\substack{\text{arbitraryly small} \\ \text{for } n \text{ suff. large}}} + \underbrace{\|A_n x - A_m x\|}_{\leq \underbrace{\|A_n - A_m\|}_{\leq \frac{\varepsilon}{2} \text{ if } n,m \geq N(\varepsilon)} \cdot \|x\|}$$

$\overset{n \to \infty}{\Rightarrow} \|Ax - A_m x\| \leq \varepsilon \|x\|$ if $m \geq N(\varepsilon)$ (indep. of x)

$$\Rightarrow \|Ax\| \leq \underbrace{\|A_m x\|}_{\leq \|A_m\| \cdot \|x\|} + \underbrace{\|Ax - A_m x\|}_{\leq \varepsilon \|x\|}$$

$\Rightarrow \|A\| \leq \|A_m\| + \varepsilon$, i.e. $A \in B(\mathcal{H})$

and $\|(A - A_m)x\| \leq \varepsilon \|x\|$

i.e. $\|A - A_m\| \leq \varepsilon$ if $m \geq N(\varepsilon)$

$\Rightarrow A_m \to A$

4) $\|ABx\| \leq \|A\| \, \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\|$

$\Rightarrow \|AB\| \leq \|A\| \cdot \|B\|$

$\Rightarrow A \cdot B \in B(\mathcal{H})$ $\square$

<u>25. Examples:</u> 1) $\mathcal{H} = \mathbb{F}^n$, $(e_i)_{i=1}^n$ ONB of $\mathcal{H}$

$A \in B(\mathcal{H}) \Rightarrow Ae_k = \sum_{l=1}^n \alpha_{kl} e_l$

$\Rightarrow A \mathrel{\hat{=}} (\alpha_{kl})_{k,l=1}^n$

$B(\mathcal{H}) \mathrel{\hat{=}} M_n(\mathbb{F})$ $n \times n$ - matrices

2) $\mathcal{H} = L^2(a,b) = \{ f : [a,b] \to \mathbb{F} \mid \int_a^b |f(t)|^2 \, dt < \infty \}$

Consider

$k \in L^2([a,b] \times [a,b])$, i.e. $\int_a^b \int_a^b |k(s,t)|^2 \, ds \, dt < \infty$

and define

$K : L^2(a,b) \to L^2(a,b)$ by

$(Kf)(s) := \int_a^b k(s,t) f(t) \, dt$

Then: $K \in B(L^2(a,b))$ and $\|K\| \leq \|k\|_{L^2([a,b] \times [a,b])}$

because:

$\|Kf\|^2 = \int |(Kf)(s)|^2 \, ds$

$= \int \underbrace{\left| \int k(s,t) f(t) \, dt \right|^2}_{\overset{C-S}{\leq} \int |k(s,t)|^2 dt \cdot \int |f(t)|^2 dt} ds$

$$\Rightarrow \|kf\|^2 \le \underbrace{\iint |k(s,t)|^2 \, dt\, ds}_{\|k\|^2_{L^2([a,b]\times[a,b])}} \cdot \underbrace{\int |f(t)|^2 \, dt}_{\|f\|^2_{L^2[a,b]}}$$

k is called <u>integral operator</u> (or Hilbert-Schmidt operator) with kernel k.

note: kernels $\hat{=}$ continuous analogues of matrices

but: not each bounded operator is integral operator

e.g. $\dim \mathcal{H} = \infty \Rightarrow id$ is not an integral operator

3) special integral operator

$\mathcal{H} = L^2(0,1)$

$$k(s,t) = \begin{cases} 0 & s \le t \\ 1 & s > t \end{cases}$$

$$(Vf)(s) = \int_0^s f(t)\, dt \qquad (0 \le s \le 1)$$

V is called Volterra-operator

$\|V\| = \frac{2}{\pi}$ non trivial

(by (2), we know $\|V\| \le \frac{1}{\sqrt{2}}$)

<u>2.6. Theorem:</u> Let $A \in B(\mathcal{H})$. There exists a unique operator $A^* \in B(\mathcal{H})$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \forall x, y \in \mathcal{H}$$

<u>2.7. Def.:</u> A^* is called the <u>adjoint</u> of A.

<u>Proof:</u> Fix $y \in \mathcal{H}$; consider

$$L: \mathcal{H} \to \mathbb{F}$$
$$x \mapsto \langle Ax, y\rangle$$

then: - L linear

- L bounded: $|Lx| = |\langle Ax, y\rangle|$

$$\leq \|Ax\| \cdot \|y\|$$
$$\leq \|A\| \, \|x\| \, \|y\|$$

$$\Rightarrow L \in \mathcal{H}^*, \text{ and } \|L\| \leq \|A\| \, \|y\|$$

$\overset{\text{Riesz}}{\underset{1.25}{\Rightarrow}}$ $\exists!\, y_0 \in \mathcal{H}$ s.th. $L(x) = \langle x, y_0\rangle$

$$\overset{\|}{\langle Ax, y\rangle}$$

Define $A^*: \mathcal{H} \to \mathcal{H}$ by $A^* y = y_0$

then: - A^* linear

- A^* bounded: $\|y_0\| \overset{1.25}{=} \|L\| \leq \|A\| \, \|y\|$

$$\overset{\|}{\|A^* y\|}$$

$$\Rightarrow \|A^*\| \leq \|A\| \Rightarrow A^* \in B(\mathcal{H}) \quad \square$$