<u>2.8. Prop.:</u> If $A, B \in B(\mathcal{H})$ and $\alpha, \beta \in \mathbb{F}$, then

1) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$

2) $(AB)^* = B^* A^*$

3) $A^{**} = A$

4) If A is invertible in $B(\mathcal{H})$
(i.e. $\exists A^{-1} \in B(\mathcal{H})$ s.th. $AA^{-1} = A^{-1}A = \mathrm{id}$),
then A^* is invertible and
$(A^*)^{-1} = (A^{-1})^*$

5) $\|A\| = \|A^*\|$

6) $\|A\|^2 = \|AA^*\|$

<u>2.9. Remarks:</u> 1) Banach algebra + (1), .., (5)
= Banach $*$-algebra

Banach $*$-algebra + (6) = C^*-algebra

2) $\mathcal{H} = \mathbb{C}^n$, then $B(\mathbb{C}^n) \hat{=} M_n(\mathbb{C})$ and

$$A = (a_{ij})_{i,j=1}^n \Rightarrow A^* = (\bar{a}_{ji})_{i,j=1}^n$$

3) $\mathcal{H} = L^2(a,b)$

K integral operator with kernel k

$\Rightarrow K^*$ —— " —— k^*

where $k^*(s,t) = \overline{k(t,s)}$

<u>Proof</u>: 1) – 4) direct checking!

e.g.: $\langle (\alpha A + \beta B) x, y \rangle = \langle x, (\alpha A + \beta B)^* y \rangle$

$$\shortparallel$$

$$\alpha \langle A x, y \rangle + \beta \langle B x, y \rangle = \alpha \langle x, A^* y \rangle + \beta \langle x, B^* y \rangle$$
$$= \langle x, (\bar{\alpha} A^* + \bar{\beta} B^*) y \rangle$$

5) $\left.\begin{array}{l} \text{Proof of 2.6.} \Rightarrow \|A^*\| \leq \|A\| \\ A \rightsquigarrow A^* \Rightarrow \|A\| \underset{3)}{=} \|A^{**}\| \leq \|A^*\| \end{array}\right\} \Rightarrow \|A\| = \|A^*\|$

6) $\|Ax\|^2 = \langle Ax, Ax \rangle$

$$= \langle A^* A x, x \rangle$$
$$\overset{CS}{\leq} \|A^* A x\| \, \|x\|$$
$$\leq \|A^* A\| \, \|x\|^2$$

$\Rightarrow \|A\|^2 \leq \|A^* A\| \leq \|A^*\| \, \|A\| \overset{5)}{=} \|A\|^2$

$\Rightarrow$ every where "="

$\Rightarrow \|A^* A\| = \|A\|^2$ $\square$

<u>2.10. Prop.</u>: If $A \in B(\mathcal{H})$, then

$$\ker A = (\operatorname{ran} A^*)^\perp$$

<u>2.11. Remark</u>: note: $\ker A$, $A^\perp$ are always closed, but not $\operatorname{ran} A$ in general

thus: $\ker A^\perp = (\operatorname{ran} A^*)^{\perp\perp} = \overline{\operatorname{ran} A^*}$

Proof: "$\subset$": Consider $x \in \ker A$

to show: $x \in (\operatorname{ran} A^*)^\perp$, i.e. $x \perp y \quad \forall y = A^* z \in \operatorname{ran} A^*$

$$\langle x, y \rangle = \langle x, A^* z \rangle = \langle \underbrace{Ax}_{=0}, z \rangle = 0$$

$\Rightarrow \ker A \subset (\operatorname{ran} A^*)^\perp$

"$\supset$": Consider $x \in (\operatorname{ran} A^*)^\perp$, i.e.

$x \perp y \quad \forall y = A^* z \in \operatorname{ran} A^*$

i.e. $\langle x, A^* z \rangle = 0 \quad \forall z \in \mathcal{H}$

$\qquad \Vert$

$\langle Ax, z \rangle \quad \Rightarrow Ax = 0$

$\qquad \Rightarrow x \in \ker A$

$\Rightarrow (\operatorname{ran} A^*)^\perp \subset \ker A$ $\square$

2.12. Def.: 1) $A \in B(\mathcal{H})$ is self-adjoint (or hermitian) if $A = A^*$

2) $A \in B(\mathcal{H})$ is normal, if $AA^* = A^*A$

3) $\mathcal{U} \in B(\mathcal{H})$ is unitary if $\mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = \mathrm{id}$

4) $V \in B(\mathcal{H})$ is isometric if $V^*V = \mathrm{id}$

5) $P \in B(\mathcal{H})$ is a (orthogonal) projection if $P^* = P = P^2$

2.13. Remark: isometric means

$$\|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, V^*Vx\rangle = \langle x, x\rangle = \|x\|^2$$

if $\dim \mathcal{H} = \infty$: isometric $\not\Rightarrow$ surjective

(compare one-sided shift 1.44)

isometric + surjective $\Leftrightarrow$ unitary

2.14 Prop.: 1) Consider $A = A^* \in B(\mathcal{H})$.

i) $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$

ii) $\langle Ax, x\rangle = 0 \quad \forall x \in \mathcal{H} \Rightarrow A = 0$

2) If $\mathbb{F} = \mathbb{C}$ and $A \in B(\mathcal{H})$, then:

i) $\langle Ax, x\rangle = 0 \quad \forall x \in \mathcal{H} \Rightarrow A = 0$

ii) $A = A^* \Leftrightarrow \langle Ax, x\rangle \in \mathbb{R} \quad \forall x \in \mathcal{H}$

Proof: Exercise!