# 13. Commutative Banach Algebras

**13.1. Def.:** Let $A$ be a Banach algebra. A complex homomorphism (or character) is a homomorphism $\varphi : A \to \mathbb{C}$ with $\varphi \neq 0$, i.e.:

$\varphi$ linear, $\varphi(xy) = \varphi(x)\varphi(y)$ $\forall x, y \in A$,
$\varphi \neq 0$

**13.2. Proposition:** Let $A$ be a Banach algebra and $\varphi : A \to \mathbb{C}$ a complex homomorphism. Then we have:

i) $\varphi(1) = 1$

ii) $\varphi(x) \neq 0$ $\quad \forall x \in A$ with $x$ invertible

iii) $\|\varphi\| = 1$, in particular $\varphi \in A^*$

**Proof:** i) Consider $y \in A$ with $\varphi(y) \neq 0$ (exists, since $\varphi \neq 0$)

$\Rightarrow \varphi(y) = \varphi(y \cdot 1) = \varphi(y) \cdot \varphi(1)$

$\Rightarrow 1 = \varphi(1)$

ii) Let $x \in A$ be invertible

$\Rightarrow \varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1) = 1 \Rightarrow \varphi(x) \neq 0$

iii) Consider $x \in A$; to show: $|\varphi(x)| \leq \|x\|$

Put $\lambda := \varphi(x)$

$\Rightarrow \varphi(\lambda 1 - x) = \lambda \cdot 1 - \varphi(x) = 0$

$\overset{ii)}{\Rightarrow}$ $\lambda 1 - x$ not invertible

i.e., $\lambda \in \sigma(x) \subset \{\lambda \mid |\lambda| \leq \|x\|\}$

$\Rightarrow |\varphi(x)| = |\lambda| \leq \|x\| \quad \Rightarrow \quad \|\varphi\| \leq 1$

since $\varphi(1) = 1 \quad \Rightarrow \quad \|\varphi\| = 1$ □

13.3. Remarks: 1) In general Banach algebras there might not be many complex homomorphisms; in particular, because of

$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$

they cannot see non-commutativity.
In commutative Banach algebras, however, the complex homomorphism determine much of the structure.

2) The main information about complex homomorphisms is contained in their kernel. These are maximal ideals.

**13.4. Def.:** Let $A$ be a Banach algebra.

1) An ideal $\mathcal{J}$ in $A$ is a linear subspace of $A$ with $\emptyset \neq \mathcal{J} \neq A$ s.th.

$$ax, xa \in \mathcal{J} \qquad \forall x \in \mathcal{J}, a \in A$$

2) A maximal ideal is an ideal which is not contained in a strictly larger ideal.

**13.5. Proposition:** Let $A$ be a Banach algebra.

1) Let $\mathcal{J}$ be an ideal in $A$. Then:

i) $\mathcal{J}$ does not contain invertible elements from $A$;

ii) $\mathrm{dist}(1, \mathcal{J}) = 1$;

iii) $\overline{\mathcal{J}}$ is an ideal in $A$

iv) If $\mathcal{J}$ is maximal, then it is closed.

2) Each ideal is contained in a maximal ideal.

Proof: Let $x \in \mathcal{J}$ be invertible in $A$, i.e. $x^{-1} \in A \Rightarrow 1 = x \cdot x^{-1} \in \mathcal{J}$

$\Rightarrow a = a \cdot 1 \in \mathcal{J} \;\; \forall a \in A \Rightarrow A = \mathcal{J}$

ii) $x \in J \Rightarrow \| 1 - x \| \geq 1$

because otherwise, by 12.6.,

$x = 1 - \underbrace{(1-x)}_{\|\cdot\| < 1}$ would be invertible

$\Rightarrow \text{dist}(1, J) \geq 1$

$0 \in J \Rightarrow \text{dist}(1, J) \leq \| 1 - 0 \| = \|1\| = 1$

$\Rightarrow \text{dist}(1, J) = 1$

iii) $\text{dist}(1, J) = 1 \Rightarrow 1 \notin \bar{J}$, i.e. $\bar{J} \neq A$

$\bar{J}$ linear subspace: clear

to show: $a \in A$, $x \in \bar{J} \Rightarrow ax, xa \in \bar{J}$

$x \in \bar{J} \Rightarrow \exists x_n \in J$ with $x_n \to x$

$\Rightarrow \underset{\substack{\cap \\ J}}{a x_n} \to ax$ (since multiplication is continuous)

$\Rightarrow ax \in \bar{J}$

same for $xa$

iv) $\left. \begin{array}{l} J \subset \bar{J} \\ J \text{ maximal} \end{array} \right\} \Rightarrow J = \bar{J}$

2) follows by Zorn's Lemma as follows

Let $\mathcal{J}$ be ideal in $A$

Put $\mathcal{Z} := \left\{ I \subset A \;\middle|\; \begin{array}{l} I \text{ ideal in } A \\ \mathcal{J} \subset I \end{array} \right\}$

$\Rightarrow (\mathcal{Z}, \subset)$ is partially ordered:

Consider chain $\mathcal{K}$ in $\mathcal{Z}$, then

$I_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} I$ ideal in $A$

and $I \subset I_{\mathcal{K}} \quad \forall I \in \mathcal{K}$

thus: $I_{\mathcal{K}}$ is upper bound for $\mathcal{K}$

$\overset{\text{Zorn's}}{\underset{\text{Lemma}}{\Longrightarrow}}$ $\exists$ maximal element $I_0 \in \mathcal{Z}$

$\Rightarrow I_0$ maximal ideal in $A$ with $\mathcal{J} \subset I_0$ $\square$

<u>13.6. Proposition</u>: Let $A$ be a Banach algebra and $\mathcal{J}$ a closed ideal in $A$. Then the quotient vector space

$A/\mathcal{J} = \{a + \mathcal{J} \mid a \in A\}$, $\left[ \begin{array}{l} \text{i.e. } a+\mathcal{J} = b + \mathcal{J} \\ \Leftrightarrow a - b \in \mathcal{J} \end{array} \right]$

equipped with the operations

$\lambda(a + \mathcal{J}) := \lambda a + \mathcal{J}$

$(a+\mathcal{J}) + (b+\mathcal{J}) := (a+b) + \mathcal{J}$

$(a+\mathcal{J}) \cdot (b+\mathcal{J}) := ab + \mathcal{J}$

and the norm

$$\|a+J\| := \inf_{x \in J} \|a+x\| = \operatorname{dist}(a, J)$$

becomes a Banach algebra and the quotient map

$$\begin{aligned} \pi : A &\to A/J \\ a &\mapsto a+J \end{aligned}$$

is a homomorphism.

<u>Proof</u>: Exercise ! $\square$

<u>13.7. Theorem</u>: Let $A$ be a commutative Banach algebra and $\Sigma$ the set of all complex homomorphisms of $A$. Then:

i) Each maximal ideal of $A$ is the kernel of a $\varphi \in \Sigma$

ii) For $\varphi \in \Sigma$, the kernel of $\varphi$ is a maximal ideal of $A$.

iii) $a \in A$ invertible $\iff \varphi(a) \neq 0 \quad \forall \varphi \in \Sigma$

iv) — " — $\iff$ there is no ideal $J$ in $A$ with $a \in J$

v) $\lambda \in \sigma(a) \iff \varphi(a) = \lambda$ for some $\varphi \in \Sigma$

i.e. $\sigma(a) = \{\varphi(a) \mid \varphi \in \Sigma\}$

(vi) The correspondence $\varphi \longleftrightarrow \ker \varphi$ is one-to-one, i.e.,
$\ker \varphi_1 = \ker \varphi_2 \iff \varphi_1 = \varphi_2$

<u>Proof</u>: i) Let $\mathcal{J}$ be maximal ideal in $A$
$\overset{13.5}{\Longrightarrow}$ $\mathcal{J}$ is closed
$\overset{13.6.}{\Longrightarrow}$ $A/\mathcal{J}$ Banach algebra

we will show: $A/\mathcal{J} \cong \mathbb{C}$; then
$\pi : A \to A/\mathcal{J} \cong \mathbb{C}$ is wanted homomorphi[sm]

Consider $A/\mathcal{J} \ni \pi(a) \neq 0$, i.e. $a \in A$, $a \notin \mathcal{J}$

Put $I := \{ ba + x \mid b \in A, x \in \mathcal{J} \}$

$$\left.\begin{array}{l} I \text{ linear} \\ IA, AI \subset I \\ \mathcal{J} \subset I \quad (b=0) \\ \mathcal{J} \neq I \quad (b=1, x=0 \\ \qquad \Rightarrow a \in I) \end{array}\right\} \overset{\mathcal{J} \text{ maximal}}{\Longrightarrow} \quad I = A$$

thus: $1 \in I$, hence $\exists\, b \in A, x \in \mathcal{J}$:
$$ba + x = 1$$

$$\Rightarrow \pi(b)\pi(a) + \underbrace{\pi(x)}_{=0} = \pi(1) = 1$$

$\Rightarrow \pi(a)$ is invertible in $A/J$

hence: every element $\neq 0$ in $A/J$ is invertible

$\overset{\text{Gelfand-Mazur}}{\underset{12.10}{\Longrightarrow}} \quad A/J \cong \mathbb{C}$

$\Rightarrow \pi : A \to A/J \cong \mathbb{C}$

thus: $\pi \in \Sigma$ and $\ker \pi = J$

ii) Consider $\varphi \in \Sigma \Rightarrow J := \ker \varphi$ ideal

$[\varphi(x) = 0 \Rightarrow \varphi(ax) = \varphi(a)\varphi(x) = 0 \quad \forall a \in A$

$\varphi(1) = 1 \Rightarrow 1 \notin J$

$J$ maximal, since $\operatorname{codim} J = 1$: $A = J \oplus \mathbb{C} \cdot 1$

$(a \in A \Rightarrow a = \underbrace{(a - \varphi(a) \cdot 1)}_{\in J} + \varphi(a) \cdot 1$

iii) Let $a \in A$ be invertible

$\overset{13.2.}{\Longrightarrow} \varphi(a) \neq 0 \quad \forall \varphi \in \Sigma$

Let $a \in A$ be not invertible

$\Rightarrow J := \{ba \mid b \in A\}$ ideal

(as $1 \notin J$)

$\overset{13.5.}{\Longrightarrow} \exists$ maximal ideal $I$ with $J \subset I$

$\overset{i)}{\Longrightarrow} \exists \varphi \in \Sigma : I = \ker \varphi$

in particular $\varphi(a) = 0$ (as $a \in J \subset I$) for $b=1$

iv) Let $a$ be invertible

$\overset{13.5.}{\Longrightarrow}$ $a$ is not contained in an ideal

Let $a$ be not invertible

$\Rightarrow a \in J := \{ba \mid b \in A\}$ ideal

v) $\lambda \in \sigma(a) \Longleftrightarrow \lambda 1 - a$ not invertible

$\overset{(iii)}{\Longleftrightarrow} \exists \varphi \in \Sigma : \varphi(\lambda 1 - a) = 0$

$\Longleftrightarrow$ — " — $: \lambda = \varphi(a)$

vi) Assume $\ker \varphi_1 = \ker \varphi_2$ and consider $a \in A$, then

$a - \varphi_1(a) \cdot 1 \in \ker \varphi_1 = \ker \varphi_2$

$$\Rightarrow \underbrace{\varphi_2(a - \varphi_1(a) \cdot 1)}_{\shortparallel} = 0$$

$$\varphi_2(a) - \varphi_1(a) \cdot \varphi_2(1) = \varphi_2(a) - \varphi_1(a)$$

$\Rightarrow \varphi_2(a) = \varphi_1(a) \qquad \forall a \in A$

$\Rightarrow \varphi_2 = \varphi_1$ $\square$

13.8. Proposition: Let $A$ be a commutative $C^*$-algebra. Then we have:

i) If $a \in A$ is selfadjoint (i.e., $a = a^*$), then $\sigma(a) \subset \mathbb{R}$

ii) If $u \in A$ is unitary (i.e. $uu^* = 1 = u^*u$), then $\sigma(u) \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$

iii) $\varphi(a^*) = \overline{\varphi(a)} \quad \forall \varphi \in \Sigma, a \in A$

13.9. Remark: By 12.18, those statements are (as in 12.19) then also true for general $C^*$-algebras

Proof: i) by 13.7.(v) we have to show:

$$\varphi(a) \in \mathbb{R} \quad \forall \varphi \in \Sigma.$$

Consider a complex homomorphism $\varphi : A \to \mathbb{C}$ and write

$\varphi(a) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$

to show: $\beta = 0$

Consider now, for $t \in \mathbb{R}$, $b := a + it1$

$$\Rightarrow \varphi(b) = \varphi(a) + it\,\varphi(1)$$
$$= \alpha + i(\beta + t)$$

$$b^* b = (a - it1)(a + it1) \quad (\text{since } a^* = a)$$
$$= a^2 + t^2$$

thus $|\varphi(b)|^2 \leq \|b\|^2$ (since $\|\varphi\| = 1$)
$$= \|b^* b\|$$
$$\leq \|a^2\| + t^2$$

$$|\alpha + i(\beta + t)|^2 = \alpha^2 + (\beta + t)^2$$
$$= \alpha^2 + \beta^2 + 2\beta t + t^2$$

$\Rightarrow \alpha^2 + \beta^2 + 2\beta t \leq \|a^2\| \quad \forall t \in \mathbb{R}$

$\Rightarrow \beta = 0$

$\Rightarrow \varphi(a) \in \mathbb{R} \quad \forall \varphi \in \Sigma$

$\Rightarrow \sigma(a) \subset \mathbb{R}$

ii) to show: $\varphi(u) \in \mathbb{T} \quad \forall \varphi \in \Sigma$

Exercise!

iii) Put $x = \frac{a+a^*}{2}$ , $y = \frac{a-a^*}{2i}$ $\Rightarrow$ $x^* = x$, $y^* = y$

i.e., $a = x+iy$ , $a^* = x-iy$

$$\Rightarrow \varphi(a^*) = \varphi(x-iy)$$
$$= \varphi(x) - i\,\varphi(y)$$
$$= \overline{\varphi(x) + i\,\varphi(y)}$$ since, by i), $\varphi(x), \varphi(y) \in \mathbb{R}$
$$= \overline{\varphi(x+iy)}$$
$$= \overline{\varphi(a)}$$ $\square$