13. Commutative Banach Algebras

13.1. Def.: Let A be a Banach algebra.
A complex homomorphism (or character)
is a homomorphism $\varphi : A \to \mathbb{C}$ with
$\varphi \neq 0$, i.e.:
φ linear, $\varphi(xy) = \varphi(x)\varphi(y)$ $\forall x, y \in A$,
$\varphi \neq 0$

13.2. Proposition: Let A be a Banach algebra
and $\varphi : A \to \mathbb{C}$ a complex homomorphism.
Then we have:

i) $\varphi(1) = 1$

ii) $\varphi(x) \neq 0$ $\quad \forall x \in A$ with x invertible

iii) $\|\varphi\| = 1$, in particular $\varphi \in A^*$

Proof: i) Consider $y \in A$ with $\varphi(y) \neq 0$
(exists, since $\varphi \neq 0$)
$\Rightarrow \varphi(y) = \varphi(y \cdot 1) = \varphi(y) \cdot \varphi(1)$

$\Rightarrow 1 = \varphi(1)$

ii) Let $x \in A$ be invertible
$\Rightarrow \varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1) = 1 \Rightarrow \varphi(x) \neq 0$

iii) Consider $x \in A$; to show: $|\varphi(x)| \leq \|x\|$

Put $\lambda := \varphi(x)$

$\Rightarrow \varphi(\lambda 1 - x) = \lambda \cdot 1 - \varphi(x) = 0$

$\overset{ii)}{\Rightarrow} \lambda 1 - x$ not invertible

i.e., $\lambda \in \sigma(x) \subset \{\lambda \mid |\lambda| \leq \|x\|\}$

$\Rightarrow |\varphi(x)| = |\lambda| \leq \|x\| \qquad \Rightarrow \|\varphi\| \leq 1$

since $\varphi(1) = 1 \qquad \Rightarrow \|\varphi\| = 1$ □

13.3. Remarks: 1) In general Banach algebras there might not be many complex homomorphisms; in particular, because of

$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$

they cannot see non-commutativity.
In commutative Banach algebras, however, the complex homomorphism determine much of the structure.

2) The main information about complex homomorphisms is contained in their kernel. These are maximal ideals.

13.4. Def.: Let A be a Banach algebra.

1) An ideal $\mathcal{I}$ in A is a linear subspace of A with $\phi \neq \mathcal{I} \neq A$ s.th.

$$ax, xa \in \mathcal{I} \qquad \forall x \in \mathcal{I}, a \in A$$

2) A maximal ideal is an ideal which is not contained in a strictly larger ideal.

13.5. Proposition: Let A be a Banach algebra.

1) Let $\mathcal{I}$ be an ideal in A. Then:

i) $\mathcal{I}$ does not contain invertible elements from A;

ii) $\operatorname{dist}(1, \mathcal{I}) = 1$;

iii) $\overline{\mathcal{I}}$ is an ideal in A

iv) If $\mathcal{I}$ is maximal, then it is closed.

2) Each ideal is contained in a maximal ideal.

Proof: Let $x \in \mathcal{I}$ be invertible in A, i.e. $x^{-1} \in A \Rightarrow 1 = x \cdot x^{-1} \in \mathcal{I}$

$\Rightarrow a = a \cdot 1 \in \mathcal{I} \;\; \forall a \in A \Rightarrow A = \mathcal{I}$

ii) $x \in J \Rightarrow \|1-x\| \geq 1$

because otherwise, by 12.6.,

$x = 1 - \underbrace{(1-x)}_{\|\cdot\| < 1}$ would be invertible

$\Rightarrow \operatorname{dist}(1, J) \geq 1$

$0 \in J \Rightarrow \operatorname{dist}(1, J) \leq \|1 - 0\| = \|1\| = 1$

$\Rightarrow \operatorname{dist}(1, J) = 1$

iii) $\operatorname{dist}(1, J) = 1 \Rightarrow 1 \notin \overline{J}$, i.e. $\overline{J} \neq A$

$\overline{J}$ linear subspace : clear

to show: $a \in A$, $x \in \overline{J} \Rightarrow ax, xa \in \overline{J}$

$x \in \overline{J} \Rightarrow \exists\, x_n \in J$ with $x_n \to x$

$\Rightarrow \underset{\substack{\cap \\ J}}{a x_n} \to a x$ (since multiplication is continuous)

$\Rightarrow ax \in \overline{J}$

same for xa

iv) $\left. \begin{array}{l} J \subset \overline{J} \\ J \text{ maximal} \end{array} \right\} \Rightarrow J = \overline{J}$

2) follows by Zorn's Lemma as follows

Let $\mathcal{J}$ be ideal in A

Put $\mathcal{Z} := \left\{ I \subset A \;\middle|\; \begin{array}{l} I \text{ ideal in } A \\ \mathcal{J} \subset I \end{array} \right\}$

$\Rightarrow$ $(\mathcal{Z}, \subset)$ is partially ordered:

Consider chain $\mathcal{R}$ in $\mathcal{Z}$, then

$$I_{\mathcal{R}} := \bigcup_{I \in \mathcal{R}} I \quad \text{ideal in } A$$

and $I \subset I_{\mathcal{R}} \quad \forall I \in \mathcal{R}$

thus: $I_{\mathcal{R}}$ is upper bound for $\mathcal{R}$

$\overset{\text{Zorn's}}{\underset{\text{Lemma}}{\Longrightarrow}}$ $\exists$ maximal element $I_0 \in \mathcal{Z}$

$\Rightarrow$ I_0 maximal ideal in A with $\mathcal{J} \subset I_0$ $\square$

<u>13.6. Proposition</u>: Let A be a Banach algebra and $\mathcal{J}$ a closed ideal in A. Then the quotient vector space

$A/\mathcal{J} = \{a + \mathcal{J} \mid a \in A\}$, $\left[\begin{array}{l} \text{i.e. } a+\mathcal{J} = b+\mathcal{J} \\ \Leftrightarrow a-b \in \mathcal{J} \end{array} \right]$

equipped with the operations

$\lambda(a+\mathcal{J}) := \lambda a + \mathcal{J}$

$(a+\mathcal{J}) + (b+\mathcal{J}) := (a+b) + \mathcal{J}$

$(a+\mathcal{J}) \cdot (b+\mathcal{J}) := ab + \mathcal{J}$