10. Compact operators on Banach spaces

In the following X is a Banach space

If $\dim X = \infty$, then closed balls are not compact, and thus also

$A(\{x \mid \|x\| \le M\})$ is in general not compact for $A \in B(X)$

$\rightsquigarrow$ this makes spectral theory of $A \in B(X)$ complicated

$\rightsquigarrow$ we consider first special operators $T \in B(\mathcal{H})$ that are close to the finite-dimensional situation

10.1. Def.: Let $T : X \to X$ be linear. T is called compact if

$\overline{\{T(x) \mid \|x\| \le 1\}}$ is compact, i.e. if we have

$(x_n)_{n\in\mathbb{N}}$ bounded $\Rightarrow$ $\exists$ convergent subsequence of $(Tx_n)_{n\in\mathbb{N}}$

Notation:

$$\mathcal{K}(X) := \{T : X \to X \mid T \text{ compact}\}$$

10.2. Proposition: 1) $\mathcal{K}(X) \subset B(X)$

2) $\mathcal{K}(X)$ is closed linear subspace of $B(X)$.

3) $\mathcal{K}(X)$ is a two-sided ideal in $B(X)$, i.e.

$$\left.\begin{array}{l} A \in B(X) \\ T \in \mathcal{K}(X) \end{array}\right\} \Rightarrow AT, TA \in \mathcal{K}(X)$$

Proof: Exercise! $\square$

10.3. Def.: An operator $T: X \to X$ is of finite rank if $\operatorname{ran} T$ is finite-dimensional.

$E(X) := \{T : X \to X \mid T \text{ is of finite rank}\}$

10.4. Remarks: 1) $E(X) \subset \mathcal{K}(X) \subset B(X)$

if $\dim X = \infty$, then all inclusions are strict

2) $E(X)$ is two-sided ideal in $B(X)$, but not closed if $\dim X = \infty$

3) Hence we have

$$\overline{E(X)} \subset \mathcal{K}(X)$$

often we have "=" (in particular for Hilbert spaces),

but not always:

first counter example: Enflo 1973

concrete counter example: $X = B(\mathcal{H})$

then $\overline{E(X)} \neq \mathcal{K}(X)$ Szankowski 1981

10.5. Proposition: Let $\mathcal{H}$ be a Hilbert space.

Then $\overline{E(\mathcal{H})} = \mathcal{K}(\mathcal{H})$

Proof: Consider $T \in \mathcal{K}(\mathcal{H})$; we have to show existence of $(T_n)_n$ with

$T_n \in E(\mathcal{H})$ $\forall n \in \mathbb{N}$ and $T_n \to T$

For this we take ONB $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ (without restriction assume $\dim \mathcal{H} = \infty$, otherwise proposition is trivial)

Put $\mathcal{H}_n := \{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{C} \}$

the linear subspace spanned by $e_1, \dots, e_n$

and $P_n :=$ orthogonal projection onto $\mathcal{H}_n$

Put $T_n := P_n T \in E(\mathcal{H})$

Claim: $T_n \to T$

First we show this pointwise, i.e.

$$x \in \mathcal{H} \Rightarrow \| T_n x - Tx \| \to 0 \; ;$$

this follows by

$$\begin{aligned} T_n x &= P_n T x \\ &= \sum_{k=1}^{n} \langle Tx, e_k \rangle e_k \\ &\to \sum_{k=1}^{\infty} \langle Tx, e_k \rangle e_k = Tx \end{aligned}$$

see 1.34

compactness of T allows to improve pointwise to uniform convergence

Fix $\varepsilon > 0$, then

$\{ \mathcal{U}_\varepsilon (Tx) \mid \|x\| \le 1 \}$ is an open cover of the compact set $\overline{\{ Tx \mid \|x\| \le 1 \}}$,

hence there is a finite subcover

$\mathcal{U}_\varepsilon (Tx_1), \ldots, \mathcal{U}_\varepsilon (Tx_m)$

for finitely many x_j with $\|x_j\| \le 1$

thus: $\{T(x) \mid \|x\| \le 1\} \subseteq \bigcup_{j=1}^{m} U_\varepsilon(Tx_j)$, i.e.

$\|x\| \le 1 \Rightarrow \exists\, x_j : \|Tx - Tx_j\| < \varepsilon$

[only finitely many]

$$\Rightarrow \|Tx - T_n x\| \le \underbrace{\|Tx - Tx_j\|}_{< \varepsilon} + \underbrace{\|Tx_j - T_n x_j\|}_{< \varepsilon} + \underbrace{\|T_n x_j - T_n x\|}_{\substack{= \|P_n(Tx_j - Tx)\| \\ \le \|Tx_j - Tx\| \\ < \varepsilon}}$$

for n sufficiently large (this is independent of x, since we need estimates only for the finitely many x_j)

$< 3\varepsilon$ for n sufficiently large $\forall \|x\| \le 1$

$\Rightarrow \|T - T_n\| < 3\varepsilon$ — " — for all $\varepsilon > 0$

$\Rightarrow T_n \to T$ $\square$

10.6 Proposition: Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2(a,b)$ and $k \in L^2([a,b]\times[a,b])$. The integral operator K with kernel k, i.e.

$$(Kf)(s) = \int_a^b k(s,t)\, f(t)\, dt$$

is a compact operator.

Proof: We use (check!)

$$\left.\begin{array}{l}(e_n)_{n\in\mathbb{N}} \text{ ONB} \\ \text{of } L^2(a,b)\end{array}\right\} \Rightarrow \begin{array}{l}(e_{n,m})_{n,m\in\mathbb{N}} \text{ ONB of} \\ L^2([a,b]\times[a,b]), \text{ where}\end{array}$$

$$e_{n,m}(s,t) = e_n(s)\,\overline{e_m(t)}$$

thus: $k \in L^2(..) \Rightarrow k = \sum_{n,m=1}^{\infty} \alpha_{nm}\, e_{n,m}$

Then we put $k_N := \sum_{n,m=1}^{N} \alpha_{nm}\, e_{n,m}$

$\Rightarrow \| k - k_N \|_{L^2} \xrightarrow{N\to\infty} 0$

Let K_N be integral operator with kernel k_N

$\Rightarrow K - K_N$ has kernel $k - k_N$

$\overset{2.5}{\Rightarrow} \| K - K_N \| \leq \| k - k_N \|_{L^2} \to 0$

i.e. $K_N \to K$

remains to show: K_N has finite rank!

$$(K_N f)(s) = \int k_N(s,t) f(t)\, dt$$

$$= \sum_{n,m=1}^{N} \alpha_{nm} \int e_n(s) \overline{e_m(t)} f(t)\, dt$$

$$= \sum_{n=1}^{N} e_n(s) \left\{ \sum_{m=1}^{N} \alpha_{nm} \langle f, e_m \rangle \right\}$$

$\Rightarrow K_N f \in \operatorname{span}\{e_1, \dots, e_N\} \quad \forall f \in L^2(a,b)$

$\Rightarrow K_N \in E(L^2(a,b)) \quad \forall N$

$\Rightarrow K \in \mathcal{K}(L^2(a,b))$

<u>10.7. Remark:</u> One can consider integral operators also on other function spaces. Typically they are then compact, too.

Consider $X = C[0,1]$ and

$k \in C([0,1] \times [0,1])$.

Define integral operator $K : C[0,1] \to C[0,1]$ by

$$(Kf)(s) = \int_0^1 k(s,t) f(t)\, dt$$

Then K is a compact operator,

$K \in \mathcal{K}(X)$

Proof: Exercise!

10.8. Theorem (Schauder): Consider $T \in B(X)$. Then the following are equivalent:

(a) $T : X \to X$ is compact

(b) $T^* : X^* \to X^*$ is compact

Proof: 1) If $X = \mathcal{H}$ is Hilbert space then the proof is easy.

(a) $\Rightarrow$ (b) $T \in \mathcal{K}(\mathcal{H})$

$\Rightarrow \exists\, T_n \in E(\mathcal{H})$ s.th. $T_n \to T$

$\Rightarrow T_n^* \to T^*$ (since
$\|T_n^* - T^*\| = \|T_n - T\| \to 0$)

and $T_n^* \in E(\mathcal{H})$

(since $T_n = P_n T_n$, where P_n projection onto $\operatorname{ran} T_n$

$\Rightarrow T_n^* = T_n^* P_n^* = T_n^* P_n \in E(\mathcal{H})$)

$\Rightarrow T^* \in \mathcal{K}(\mathcal{H})$

(b) $\Rightarrow$ (a) : replace T by T^*

2) proof for general X is more complicated and relies on Arzela–Ascoli theorem

we omit the proof.