10. Compact operators on Banach spaces

In the following X is a Banach space

If $\dim X = \infty$, then closed balls are not compact, and thus also

$\overline{A(\{x \mid \|x\| \leq M\})}$ is in general not compact for $A \in B(X)$

⇝ this makes spectral theory of $A \in B(X)$ complicated

⇝ we consider first special operators $T \in B(\mathcal{H})$ that are close to the finite-dimensional situation

10.1. Def.: Let $T : X \to X$ be linear. T is called <u>compact</u> if

$\overline{\{T(x) \mid \|x\| \leq 1\}}$ is compact, i.e. if we have

$(x_n)_{n\in\mathbb{N}}$ bounded $\Rightarrow$ $\exists$ convergent subsequence of $(Tx_n)_{n\in\mathbb{N}}$

Notation:

$$\mathcal{K}(X) := \{T : X \to X \mid T \text{ compact}\}$$

10.2. Proposition: 1) $\mathcal{K}(X) \subset B(X)$

2) $\mathcal{K}(X)$ is closed linear subspace of $B(X)$.

3) $\mathcal{K}(X)$ is a two-sided ideal in $B(X)$, i.e.

$$\left.\begin{array}{l} A \in B(X) \\ T \in \mathcal{K}(X) \end{array}\right\} \Rightarrow AT, TA \in \mathcal{K}(X)$$

Proof: Exercise! $\square$

10.3. Def.: An operator $T: X \to X$ is of finite rank if $\operatorname{ran} T$ is finite-dimensional.

$E(X) := \{T : X \to X \mid T \text{ is of finite rank}\}$

10.4. Remarks: 1) $E(X) \subset \mathcal{K}(X) \subset B(X)$

if $\dim X = \infty$, then all inclusions are strict

2) $E(X)$ is two-sided ideal in $B(X)$, but not closed if $\dim X = \infty$

3) Hence we have

$$\overline{E(X)} \subset \mathcal{K}(X)$$

often we have "=" (in particular for Hilbert spaces),

but not always:

first counter example: Enflo 1973

concrete counter example: $X = B(\mathcal{H})$

then $\overline{E(X)} \neq \mathcal{K}(X)$ Szankowski 1981

<u>10.5. Proposition:</u> Let $\mathcal{H}$ be a Hilbert space.

Then $\overline{E(\mathcal{H})} = \mathcal{K}(\mathcal{H})$

<u>Proof:</u> Consider $T \in \mathcal{K}(\mathcal{H})$; we have to show existence of $(T_n)_n$ with

$T_n \in E(\mathcal{H})$ $\forall n \in \mathbb{N}$ and $T_n \to T$

For this we take ONB $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$

(without restriction assume $\dim \mathcal{H} = \infty$, otherwise proposition is trivial)

Put $\mathcal{H}_n := \{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in \mathbb{C} \}$

the linear subspace spanned by $e_1, \ldots, e_n$

and $P_n :=$ orthogonal projection onto $\mathcal{H}_n$

Put $T_n := P_n T \in E(\mathcal{H})$

Claim: $T_n \to T$

First we show this pointwise, i.e.

$$x \in \mathcal{H} \Rightarrow \| T_n x - T x \| \to 0 ;$$

this follows by

$$\begin{aligned} T_n x &= P_n T x \\ &= \sum_{k=1}^{n} \langle Tx, e_k \rangle e_k \\ &\to \sum_{k=1}^{\infty} \langle Tx, e_k \rangle e_k = Tx \end{aligned}$$

see 1.34

compactness of T allows to improve pointwise to uniform convergence

Fix $\varepsilon > 0$, then

$\{ \mathcal{U}_\varepsilon (Tx) \mid \|x\| \le 1 \}$ is an open cover of the compact set $\overline{\{ Tx \mid \|x\| \le 1 \}}$,

hence there is a finite subcover

$\mathcal{U}_\varepsilon (Tx_1), \dots, \mathcal{U}_\varepsilon (Tx_m)$

for finitely many x_j with $\|x_j\| \le 1$

thus: $\{T(x) \mid \|x\| \leq 1\} \subseteq \bigcup_{j=1}^{m} U_\varepsilon(Tx_j)$, i.e.

$\|x\| \leq 1 \Rightarrow \exists\, x_j : \|Tx - Tx_j\| < \varepsilon$

only finitely many

$$\Rightarrow \|Tx - T_n x\| \leq \underbrace{\|Tx - Tx_j\|}_{< \varepsilon} + \underbrace{\|Tx_j - T_n x_j\|}_{< \varepsilon} + \underbrace{\|T_n x_j - T_n x\|}_{\substack{= \|P_n(Tx_j - Tx)\| \\ \leq \|Tx_j - Tx\| \\ < \varepsilon}}$$

for n sufficiently large (this is independent of x, since we need estimates only for the finitely many x_j)

$< 3\varepsilon$ for n sufficiently large $\forall \|x\| \leq 1$

$\Rightarrow \|T - T_n\| < 3\varepsilon$ — " — for all $\varepsilon > 0$

$\Rightarrow T_n \to T$

□