<u>10.6 Proposition:</u> Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2(a,b)$ and $k \in L^2([a,b]\times[a,b])$. The integral operator K with kernel k, i.e.

$$(Kf)(s) = \int_a^b k(s,t)\, f(t)\, dt$$

is a compact operator.

<u>Proof:</u> We use (check!)

$$\left.\begin{array}{l}(e_n)_{n\in\mathbb{N}} \text{ ONB} \\ \text{of } L^2(a,b)\end{array}\right\} \Rightarrow \begin{array}{l}(e_{n,m})_{n,m\in\mathbb{N}} \text{ ONB of} \\ L^2([a,b]\times[a,b]), \text{ where} \\ e_{n,m}(s,t) = e_n(s)\overline{e_m(t)}\end{array}$$

thus: $k \in L^2(..) \Rightarrow k = \sum_{n,m=1}^{\infty} \alpha_{nm} e_{n,m}$

Then we put $k_N := \sum_{n,m=1}^{N} \alpha_{nm} e_{n,m}$

$$\Rightarrow \|k - k_N\|_{L^2} \xrightarrow{N\to\infty} 0$$

Let K_N be integral operator with kernel k_N

$\Rightarrow K - K_N$ has kernel $k - k_N$

$$\overset{2.5}{\Rightarrow} \|K - K_N\| \le \|k - k_N\|_{L^2} \to 0$$

i.e. $K_N \to K$

remains to show: K_N has finite rank!

$$(K_N f)(s) = \int k_N(s,t) f(t)\, dt$$

$$= \sum_{n,m=1}^{N} \alpha_{nm} \int e_n(s) \overline{e_m(t)} f(t)\, dt$$

$$= \sum_{n=1}^{N} e_n(s) \left\{ \sum_{m=1}^{N} \alpha_{nm} \langle f, e_m \rangle \right\}$$

$\Rightarrow K_N f \in \operatorname{span}\{e_1, \dots, e_N\} \quad \forall f \in L^2(a,b)$

$\Rightarrow K_N \in \mathcal{E}(L^2(a,b)) \quad \forall N$

$\Rightarrow K \in \mathcal{K}(L^2(a,b))$

<u>10.7. Remark:</u> One can consider integral operators also on other function spaces. Typically they are then compact, too. Consider $X = C[0,1]$ and $k \in C([0,1] \times [0,1])$.

Define integral operator $K : C[0,1] \to C[0,1]$ by

$$(Kf)(s) = \int_0^1 k(s,t) f(t)\, dt$$

Then K is a compact operator, $K \in \mathcal{K}(X)$

Proof: Exercise!

10.8. Theorem (Schauder): Consider $T \in B(X)$. Then the following are equivalent:

(a) $T : X \to X$ is compact

(b) $T^* : X^* \to X^*$ is compact

Proof: 1) If $X = \mathcal{H}$ is Hilbert space then the proof is easy.

(a) $\Rightarrow$ (b) $T \in \mathcal{K}(\mathcal{H})$

$\Rightarrow \exists\, T_n \in E(\mathcal{H})$ s.th. $T_n \to T$

$\Rightarrow T_n^* \to T^*$ (since $\|T_n^* - T^*\| = \|T_n - T\| \to 0$)

and $T_n^* \in E(\mathcal{H})$

(since $T_n = P_n T_n$, where P_n projection onto ran T_n

$\Rightarrow T_n^* = T_n^* P_n^* = T_n^* P_n \in E(\mathcal{H})$)

$\Rightarrow T^* \in \mathcal{K}(\mathcal{H})$

(b) $\Rightarrow$ (a): replace T by T^*

2) proof for general X is more complicated and relies on Arzela-Ascoli theorem

we omit the proof.