14. The Gelfand Transform

14.1. Example: Let K be compact space and $A = C(K)$. Then

① φ complex homomorphism

$$\iff \exists t \in K : \varphi(f) = f(t) \quad \forall f \in C(K)$$

② J maximal ideal

$$\iff \exists t \in K : J = \{ f \in C(K) \mid f(t) = 0 \}$$

$$① \underset{J = \ker \varphi}{\longleftrightarrow} ②$$

① $\Rightarrow$ ② : $J = \ker \varphi$

$$= \{ f \in C(K) \mid \underbrace{\varphi(f)}_{f(t)} = 0 \}$$

② $\Rightarrow$ ① $\ker \varphi = \{ f \mid f(t) = 0 \}$

Consider $f \in C(K)$

$\Rightarrow f - \varphi(f) \cdot 1 \in \ker \varphi$

$\Rightarrow \underbrace{(f - \varphi(f) \cdot 1)(t)}_{f(t) - \varphi(f)} = 0 \qquad \Rightarrow \varphi(f) = f(t)$

"Proof" of ①: $\varphi \in \Sigma \subset \underbrace{C(K)^*}_{\overset{?}{=}\ \text{complex measures on } K}$

i.e., $\varphi(f) = \int f(s)\, d\mu(s)$

one shows then:

φ multiplicative $\Rightarrow$ μ is δ-measure,

i.e., $\exists t \in K : \mu = \delta_t$

$\Rightarrow \varphi(f) = \int f(s)\, d\delta_t(s)$

$= f(t)$ □

Proof of ②: $J_t := \{ f \in C(K) \mid f(t) = 0 \}$

is clearly maximal ideal

to show: J maximal ideal $\Rightarrow \exists t \in K : J = J_t$

assume: $\forall t \in K\ \exists f_t \in J : f_t \notin J_t$,

i.e. $f_t(t) \neq 0$

$\Rightarrow \forall t \in K\ \exists$ neighbarhood U_t and $f_t \in J$,

$f_t(s) \neq 0 \quad \forall s \in U_t$

$\Rightarrow (U_t)_{t \in K}$ is open cover of K

K compact $\Rightarrow$ $\exists$ finite subcover, i.e.,

$\exists\ f_{t_1}, \dots, f_{t_n} \in J$ s.th. $\forall s \in K$:

$f_{t_i}(s) \neq 0$ for at least one $i = 1, \dots, n$

Consider now

$$f := \sum_{i=1}^{n} f_{t_i} \overline{f_{t_i}} \in J$$

$\Rightarrow f(s) \neq 0 \quad \forall s \in K$

$\Rightarrow f$ invertible in $C(K)$

[with $f^{-1}(s) = \frac{1}{f(s)}$]

$\Rightarrow J$ contains invertible element, contradiction to 13.5

$\Rightarrow \exists\, t \in K : J = J_t$ $\square$

<u>Conclusion</u>: In the case $A = C(K)$ we can recover K abstractly as

$$\Sigma_{C(K)} \hat{=} K$$

$$\varphi_t \longleftarrow t$$

where $\varphi_t(f) = f(t)$

thus: $t \hat{=} \varphi_t$

and

$f(t) = \varphi_t(f)$

$= \hat{f}(\varphi_t)$

This identifies

$$k \hat{=} \Sigma_{\mathbb{C}(k)} \quad \text{as sets}$$

But how about the topology?

<u>14.2. Lemma</u>: Let A be a commutative Banach algebra and Σ the set of all complex homomorphisms on A. Then there exists exactly one topology τ on Σ s.th.

i) (Σ, τ) is compact

ii) The mappings

$$\begin{aligned} \hat{a} : \Sigma &\to \mathbb{C} \\ \varphi &\mapsto \hat{a}(\varphi) = \varphi(a) \end{aligned}$$

are τ-continuous, for all $a \in A$.

<u>14.3. Reminder</u>: 1) Let X be a set and $\tau \subset P(X) = \{Y \mid Y \subset X\}$.

τ is a <u>topology</u> on X (and $U \in \tau$ are <u>open</u>), if

- $\emptyset, X \in \tau$
- arbitrary unions of open sets are open
- finite intersections of open sets are open

2) The topology is <u>Hausdorff</u> if
$\forall x, y \in X$ s.th. $x \neq y$ $\exists\, U_x, U_y \in \tau$:

$$U_x \cap U_y = \emptyset \text{ and } x \in U_x,\ y \in U_y$$

3) $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is <u>continuous</u>, if

$$f^{-1}(U) \in \tau_1 \quad \forall\, U \in \tau_2$$

4) Let $(X_i, \tau_i)_{i \in I}$ be topological spaces. Consider the product set

$$X := \prod_{i \in I} X_i$$

, i.e. $x \in X$ are of the form $x = (x_i)_{i \in I}$ with $x_i \in X_i$

Then there is a canonical <u>product topology</u> τ on X s.th. all projections ($j \in I$)

$$\pi_j : X \longrightarrow X_j$$
$$(x_i)_{i \in I} \longmapsto x_j$$

are continuous.

τ is the smallest topology with this property.

More explicitly, this τ can be described as follows:

$$\mathcal{U} \in \tau \iff \mathcal{U} = \bigcup \mathcal{U}_i \quad \text{with } \mathcal{U}_i \in \mathcal{B}$$

where

$$\mathcal{B} = \{ \prod_{i \in I} V_i \mid V_i \in \tau_i \quad \forall\, i \in I, \quad V_i \neq X_i \text{ for only finitely many } i \in I \}$$

5) One has the <u>Theorem of Tychonoff</u>:

all τ_i compact $\Rightarrow$ τ is compact

<u>Proof of 14.2.</u>: We only show: There exists a (canonical) topology on Σ with these properties!

For $a \in k$ put $k_a := \{ z \in \mathbb{C} \mid |z| \leq \|a\| \}$

$\Rightarrow k_a$ is compact in $\mathbb{C}$

$\overset{\text{Tychonoff}}{\Rightarrow}$ $X := \prod_{a \in A} k_a$ compact (w.r.t. product topology τ_X)

Consider now

$$T : \Sigma \to X$$
$$\varphi \mapsto (\varphi(a))_{a \in A}$$

(note that, by 13.2, $|\varphi(a)| \leq \|a\|$)

T is clearly injective and

$$\operatorname{ran} T = \{ (x_a)_{a\in A} \mid x_1 = 1,\ x_{ab} = x_a x_b,\ x_{\frac{a+b}{2}} = \tfrac{1}{2}(x_a + x_b),\ x_{\lambda a} = \lambda x_a \quad \forall a,b \in A,\ \lambda \in \mathbb{C},\ |\lambda| \leq 1 \}$$

$\subset X$ closed in X

(since defined by equalities of continuous functions)

$\Rightarrow$ $\operatorname{ran} T$ is compact w.r.t. τ_X

We identify now Σ with $\operatorname{ran} T \subset X$, this gives then topology τ on Σ (via $\mathcal{U} \in \tau \Leftrightarrow \mathcal{U} = T^{-1}(V)$ for $V \in \tau_X$)

$\Rightarrow$ τ compact

Furthermore,

$\hat{a} : \Sigma \longrightarrow \mathbb{C}$, $\varphi \longmapsto \varphi(a)$ corresponds to $T(\Sigma) \longrightarrow \mathbb{C}$ $\quad (\varphi(b))_{b\in A} \longmapsto \varphi(a)$

thus: $\hat{a}$ corresponds to the projection π_a in the product space and is thus continuous

<u>14.4. Def</u>: Let A be a commutative Banach algebra.

1) The set Σ of all complex homomorphisms on A, equipped with the canonical topology from 14.2., is the <u>maximal ideal space</u> or <u>spectrum</u> of A.

2) The map

$$\wedge : A \to C(\Sigma)$$
$$a \mapsto \hat{a}$$

where $\hat{a}(\varphi) = \varphi(a)$

is the <u>Gelfand representation</u> of A; and $\hat{a}$ is the <u>Gelfand transform</u> of a.

<u>14.5. Theorem</u>: Let A be a commutative Banach algebra. Then the Gelfand representation is an algebra homomorphism and we have: i) $\hat{a}(\Sigma) = \sigma(a)$ $\forall a \in A$

ii) $\|\hat{a}\| = r(a) \leq \|a\|$ $\forall a \in A$

iii) $\varphi_1, \varphi_2 \in \Sigma$ with $\varphi_1 \neq \varphi_2$

$$\Rightarrow \exists a \in A : \hat{a}(\varphi_1) \neq \hat{a}(\varphi_2)$$

Proof: Homomorphismus clear, since

$$(\lambda a+\mu b)^\wedge(\varphi) = \varphi(\lambda a+\mu b)$$
$$= \lambda\,\varphi(a)+\mu\,\varphi(b)$$
$$= \lambda\,\hat a(\varphi)+\mu\,\hat b(\varphi)$$

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \hat a(\varphi)\,\hat b(\varphi)$$

$$\hat{1}(\varphi) = \varphi(1) = 1$$

i) $\hat a(\Sigma) = \{\underbrace{\hat a(\varphi)}_{\varphi(a)} \mid \varphi\in\Sigma\} \underset{13.7(v)}{=} \sigma(a)$

ii) $\|\hat a\| = \sup\limits_{\varphi\in\Sigma} |\hat a(\varphi)|$

$$= \sup_{\varphi\in\Sigma}|\varphi(a)| = \sup_{\lambda\in\sigma(a)}\lambda = r(a) \underset{12.9.}{\le} \|a\|$$

iii) Assume $\underbrace{\hat a(\varphi_1)}_{\varphi_1(a)} = \underbrace{\hat a(\varphi_2)}_{\varphi_2(a)} \quad \forall a\in A$

$\Rightarrow \varphi_1 = \varphi_2$

$\square$

14.6 Theorem (Gelfand - Naimark):

Let A be a commutative C^*-algebra. Then the Gelfand representation $\hat{\ }: A \to C(\Sigma)$ is an isometric $*$-isomorphism.

Proof: each $a \in A$ is normal

$\overset{12.13}{\Longrightarrow}$ $r(a) = \|a\| \quad \forall a \in A$

$\Longrightarrow \|\hat{a}\| = r(a) = \|a\| \quad \forall a \in A$

i.e., $\hat{\ }$ is isometric

$\Rightarrow \hat{A}$ is a closed subalgebra of $C(\Sigma)$

14.5(iii) $\Longrightarrow \hat{A}$ separates points of Σ

$1 = \hat{1} \in \hat{A} \Longrightarrow \hat{A}$ unital subalgebra

$\widehat{a^*}(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{\hat{a}(\varphi)}$

$$\Longrightarrow \widehat{a^*} = \overline{\hat{a}}$$

thus: $\hat{a} \in \hat{A} \Longrightarrow \overline{\hat{a}} \in \hat{A}$

$\overset{\text{Stone-}}{\underset{\text{Weierstraß}}{\Longrightarrow}}$ $\hat{A} = C(\Sigma)$

<u>14.7. Proposition:</u> Let A be a C^*-algebra and $a \in A$ normal, i.e., $aa^* = a^*a$. Put

$$C^*(a) := \overline{\{\text{polynomials in } a \text{ and } a^*\}}^{\|\cdot\|},$$

the smallest C^*-algebra that contains a (and thus also a^*). Then $C^*(a)$ is commutative, thus

$$C^*(a) \mathrel{\hat{=}} C(\Sigma)$$

where $\Sigma = \Sigma_{C^*(a)}$ is the spectrum of $C^*(a)$. Furthermore,

$$\Sigma_{C^*(a)} \mathrel{\hat{=}} \sigma(a) \quad \text{as topological spaces}$$

via the homeomorphism

$$\hat{a} : \Sigma_{C^*(a)} \to \sigma(a)$$

<u>Proof:</u> $\hat{a} \in C(\Sigma)$, i.e.,

$$\hat{a} : \Sigma \to \mathbb{C} \quad \text{continuous}$$

$$\operatorname{ran} \hat{a} = \{\hat{a}(\varphi) \mid \varphi \in \Sigma\} = \{\lambda \mid \lambda \in \sigma(a)\}$$

$$\Rightarrow \hat{a} : \Sigma \to \sigma(a) \quad \text{surjective}$$

Assume $\hat{a}(\varphi_1) = \hat{a}(\varphi_2)$, i.e.

$$\varphi_1(a) = \varphi_2(a)$$

$$\overset{13.8.}{\Longrightarrow} \varphi_1(a^*) = \overline{\varphi_1(a)} = \overline{\varphi_2(a)} = \varphi_2(a^*)$$

$$\Rightarrow \varphi_1(p(a,a^*)) = \varphi_2(p(a,a^*))$$

$\forall$ polynomials $p(\cdot,\cdot)$ in two commuting variables

$$\left.\begin{array}{l} \text{such polynomials } p(a,a^*) \\ \text{are dense in } C^*(a) \\ \varphi_1, \varphi_2 \text{ continuous} \\ (\text{as } \|\varphi_1\| = 1 = \|\varphi_2\|) \end{array}\right\} \Rightarrow \begin{array}{l} \varphi_1(x) = \varphi_2(x) \\ \forall x \in C^*(a) \end{array}$$

thus: $\varphi_1 = \varphi_2$

$\Rightarrow \hat{a}$ injective

$\Rightarrow \hat{a} : \Sigma_1 \to \sigma(a)$ bijective

and so: $\hat{a}$ is continuous bijection between compact spaces and thus automatically a homeomorphism (i.e. $\hat{a}^{-1}$ is also continuous) $\square$

14.9 Theorem (continuous functional calculus for normal elements)

Let A be a C^*-algebra and $a \in A$ normal.

1) There exists exactly one $*$-homomorphism

$$\Phi : C(\sigma(a)) \to A$$
$$f \mapsto \Phi(f)$$

s.th. $\Phi(z) = a$, where

$$z : \sigma(a) \to \mathbb{C}$$
$$\lambda \mapsto \lambda$$

We will also denote $\Phi(f)$ by $f(a)$.

2) Φ is an isometry, i.e.,

$$\|f(a)\| = \|f\|_{C(\sigma(a))} = \sup_{\lambda \in \sigma(a)} |f(\lambda)|$$

and

$$\operatorname{ran} \Phi = C^*(a)$$

is the smallest C^*-algebra that contains a. [Note: $C^*(a)$ is commutative.]

Proof: i) Uniqueness

Let Φ be with properties as in (1)

$\Rightarrow \Phi(p)$ is determined for any polynomial p in z and $\bar{z}$

$$p(z) = \sum \alpha_{nm} z^n \bar{z}^m \Rightarrow \Phi(p) = p(a) = \sum \alpha_{nm} a^n a^{*m}$$

Since (Exercise!)

$p_n \longrightarrow f$ in $C(\sigma(a))$ $\Rightarrow$ $\Phi(p_n) \longrightarrow \Phi(f)$ in A

and, by Stone-Weierstraß, polynomials $p(z,\bar{z})$ are dense in $C(\sigma(a))$

$\Rightarrow \Phi$ is determined on all of $C(\sigma(a))$!

ii) Existence

We have the following identifications

$$\wedge : C^*(a) \longrightarrow C(\Sigma_{C^*(a)}) \qquad \text{Gelfand isomorph.}$$
$$y \longmapsto \hat{y}$$

and via

$$\hat{a} : \Sigma_{C^*(a)} \longrightarrow \sigma(a)$$

also

$$\begin{aligned} C(\sigma(a)) &\longrightarrow C(\Sigma_{C^*(a)}) \\ f &\longmapsto f \circ \hat{a} \end{aligned}$$

Thus

$$C^*(a) \xrightarrow{\ \wedge\ } C(\Sigma) \underset{\circ \hat{a}}{\overset{\circ \hat{a}^{-1}}{\rightleftarrows}} C(\sigma(a))$$

and $\Phi(f)$ is uniquely determined by

$$\widehat{\Phi(f)} = f \circ \hat{a}$$

iii) Properties of Φ

∘ isometry

$$\begin{aligned} \|\Phi(f)\|_A &= \|\widehat{\Phi(f)}\|_{C(\Sigma)} \\ &= \|f \circ \hat{a}\|_{C(\Sigma)} \\ &= \sup_{\varphi \in \Sigma} |f(\hat{a}(\varphi))| \\ &= \sup_{\lambda \in \sigma(a)} |f(\lambda)| \\ &= \|f\|_{C(\sigma(a))} \end{aligned}$$

- $\Phi(z) = a$

since : $\widehat{\Phi(z)} = z \circ \hat{a} = \hat{a}$

- $\bar{f}(a) = f(a)^*$, since

$\widehat{f(a)} = f \circ \hat{a}$

$\Rightarrow \widehat{f(a)^*} = \overline{\widehat{f(a)}} = \bar{f} \circ \hat{a}$

- homomorphism

$\widehat{(f+g)(a)} = (f+g) \circ \hat{a}$

$= f \circ \hat{a} + g \circ \hat{a}$

$= \widehat{f(a)} + \widehat{g(a)}$

$= \widehat{f(a) + g(a)}$

$\Rightarrow (f+g)(a) = f(a) + g(a)$

$(f \cdot g)(a) = f(a) \cdot g(a)$ similar

$\square$

14.10. Theorem (measurable functional calculus for normal operators)

Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$ normal. Then there exists exactly one $*$-homomorphism

$$\tilde{\Phi} : M_\infty(\sigma(A)) \to B(\mathcal{H})$$
$$h \mapsto \tilde{\Phi}(h) =: h(A)$$

s.th. $\tilde{\Phi}(z) = A$ (where $z(\lambda) = \lambda$) and

$$\left.\begin{array}{l} h_n \ (n \in \mathbb{N}),\ h \in M_\infty(\sigma(A)) \\ h_n \to h \text{ u.p.} \end{array}\right\} \Rightarrow \lim_{n\to\infty} \langle \tilde{\Phi}(h_n)x, y\rangle = \langle \tilde{\Phi}(h)x, y\rangle \quad \forall x, y \in \mathcal{H}$$

We have: $\tilde{\Phi}$ is continuous and $\|\tilde{\Phi}\| = 1$

14.11 Notations: For $K \subset \mathbb{R}$ compact we denote

i) $M_\infty(K) := \{ h : K \to \mathbb{C} \mid h \text{ measurable and bounded} \}$

[this is a commutative C^*-algebra w.r.t. $\|h\| = \sup_{t\in K} |h(t)|$ and $h^*(t) = \overline{h(t)}$

ii) $h_n \to h$ u.p. (uniformly pointwise)

iff: ∘ $h(t) = \lim_{n\to\infty} h_n(t) \quad \forall t \in K$

$$\circ \ \sup_{n\in\mathbb{N}} \underbrace{\sup_{t\in K} |h_n(t)|}_{\|h_n\|} < \infty$$

<u>14.12. Remarks:</u> 1) $M_\infty(K)$ is the smallest class M of functions $K \to \mathbb{C}$ s.th.

i) $C(K) \subset M$

ii) M is closed under u.p. convergence

2) $\tilde{\Phi}$ continuous means:

$$h_n \to h \implies \tilde{\Phi}(h_n) \to \tilde{\Phi}(h)$$

w.r.t. sup-norm

3) $\operatorname{ran} \tilde{\Phi} = vN(A)$ is the (commutative) von Neumann algebra generated by A, i.e. the smallest C^*-algebra containing A that is closed in the weak operator topology.

$$\left.\begin{array}{c} \lim \langle B_n x, y\rangle = \langle Bx, y\rangle \\ \forall\, x, y \in \mathcal{H} \\ B_n \in vN(A) \end{array}\right\} \Rightarrow B \in vN(A)$$