# 14. The Gelfand Transform

**14.1. Example:** Let $K$ be compact space and $A = C(K)$. Then

① $\varphi$ complex homomorphism

$$\iff \exists t \in K : \varphi(f) = f(t) \quad \forall f \in C(K)$$

② $J$ maximal ideal

$$\iff \exists t \in K : J = \{ f \in C(K) \mid f(t) = 0 \}$$

$$① \xleftrightarrow{J = \ker \varphi} ②$$

① $\Rightarrow$ ② : $J = \ker \varphi$

$$= \{ f \in C(K) \mid \underbrace{\varphi(f)}_{f(t)} = 0 \}$$

② $\Rightarrow$ ① $\ker \varphi = \{ f \mid f(t) = 0 \}$

Consider $f \in C(K)$

$\Rightarrow f - \varphi(f) \cdot 1 \in \ker \varphi$

$\Rightarrow (f - \varphi(f) \cdot 1)(t) = 0$

$$\| $$

$f(t) - \varphi(f)$ $\qquad \Rightarrow \varphi(f) = f(t)$

"Proof" of ①: $\varphi \in \Sigma \subset \underbrace{C(K)^*}_{\overset{?}{=}\ \text{complex measures on } K}$

i.e., $\varphi(f) = \int f(s)\, d\mu(s)$

one shows then:

$\varphi$ multiplicative $\Rightarrow$ $\mu$ is $\delta$-measure,

i.e., $\exists t \in K : \mu = \delta_t$

$\Rightarrow \varphi(f) = \int f(s)\, d\delta_t(s)$

$= f(t)$ □

Proof of ②: $J_t := \{f \in C(K) \mid f(t) = 0\}$

is clearly maximal ideal

to show: $J$ maximal ideal $\Rightarrow \exists t \in K : J = J_t$

assume: $\forall t \in K\ \exists f_t \in J : f_t \notin J_t$,

i.e. $f_t(t) \neq 0$

$\Rightarrow \forall t \in K\ \exists$ neighborhood $U_t$ and $f_t \in J$,

$f_t(s) \neq 0 \quad \forall s \in U_t$

$\Rightarrow (U_t)_{t \in K}$ is open cover of $K$

$\overset{K \text{ compact}}{\Longrightarrow}$ $\exists$ finite subcover, i.e.,

$\exists\ f_{t_1}, \dots, f_{t_n} \in J$ s.th. $\forall s \in K$:

$f_{t_i}(s) \neq 0$ for at least one $i = 1, \dots, n$

Consider now

$$f := \sum_{i=1}^{n} f_{t_i} \overline{f_{t_i}} \in J$$

$\Rightarrow f(s) \neq 0 \quad \forall s \in K$

$\Rightarrow f$ invertible in $C(K)$

[with $f^{-1}(s) = \frac{1}{f(s)}$]

$\Rightarrow J$ contains invertible element, contradiction to 13.5

$\Rightarrow \exists\, t \in K : J = J_t$ $\square$

<u>Conclusion:</u> In the case $A = C(K)$ we can recover $K$ abstractly as

$$\Sigma_{C(K)} \mathrel{\hat{=}} K$$

$$\varphi_t \leftarrow\!\shortmid\ t$$

where $\varphi_t(f) = f(t)$

thus: $t \mathrel{\hat{=}} \varphi_t$ and

$f(t) = \varphi_t(f) = \hat{f}(\varphi_t)$

This identifies

$$k \hat{=} \Sigma_{\varphi(k)} \quad \text{as sets}$$

But how about the topology?

<u>14.2. Lemma</u>: Let $A$ be a commutative Banach algebra and $\Sigma$ the set of all complex homomorphisms on $A$. Then there exists exactly one topology $\tau$ on $\Sigma$ s.th.

i) $(\Sigma, \tau)$ is compact

ii) The mappings

$$\hat{a} : \Sigma \to \mathbb{C}$$
$$\varphi \mapsto \hat{a}(\varphi) = \varphi(a)$$

are $\tau$-continuous, for all $a \in A$.

<u>14.3. Reminder</u>: 1) Let $X$ be a set and $\tau \subset P(X) = \{Y \mid Y \subset X\}$.

$\tau$ is a <u>topology</u> on $X$ (and $U \in \tau$ are <u>open</u>), if

- $\emptyset, X \in \tau$
- arbitrary unions of open sets are open
- finite intersections of open sets are open

2) The topology is <u>Hausdorff</u> if

$\forall x, y \in X$ s.th. $x \neq y$ $\exists\, \mathcal{U}_x, \mathcal{U}_y \in \tau$ :

$$\mathcal{U}_x \cap \mathcal{U}_y = \phi \text{ and } x \in \mathcal{U}_x,\ y \in \mathcal{U}_y$$

3) $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is

<u>continuous</u>, if

$$f^{-1}(\mathcal{U}) \in \tau_1 \quad \forall\, \mathcal{U} \in \tau_2$$

4) Let $(X_i, \tau_i)_{i \in I}$ be topological spaces. Consider the product set

$X := \prod_{i \in I} X_i$, i.e. $x \in X$ are of the form $x = (x_i)_{i \in I}$ with $x_i \in X_i$

Then there is a canonical <u>product topology</u> $\tau$ on $X$ s.th. all projections $(j \in I)$

$$\pi_j : X \longrightarrow X_j$$
$$(x_i)_{i \in I} \longmapsto x_j$$

are continuous.

$\tau$ is the smallest topology with this property.

More explicitly, this $\tau$ can be described as follows:

$$\mathcal{U} \in \tau \iff \mathcal{U} = \bigcup \mathcal{U}_i \quad \text{with } \mathcal{U}_i \in B$$

where

$$B = \{ \prod_{i \in I} V_i \mid V_i \in \tau_i \ \forall i \in I, \ V_i \neq X_i \text{ for only finitely many } i \in I \}$$

5) One has the <u>Theorem of Tychonoff</u>:

all $\tau_i$ compact $\Rightarrow$ $\tau$ is compact

<u>Proof of 14.2.</u>: We only show: There exists a (canonical) topology on $\Sigma$ with these properties!

For $a \in k$ put $k_a := \{ z \in \mathbb{C} \mid |z| \leq \|a\| \}$

$\Rightarrow$ $k_a$ is compact in $\mathbb{C}$

$\overset{\text{Tychonoff}}{\Longrightarrow}$ $X := \prod_{a \in A} k_a$ compact (w.r.t. product topology $\tau_X$)

Consider now

$$T : \Sigma \longrightarrow X$$
$$\varphi \longmapsto (\varphi(a))_{a \in A}$$

(note that, by 13.2, $|\varphi(a)| \leq \|a\|$)

$T$ is clearly injective and

$$\operatorname{ran} T = \{ (x_a)_{a\in A} \mid x_1 = 1,\ x_{ab} = x_a x_b,\ x_{\frac{a+b}{2}} = \tfrac{1}{2}(x_a + x_b),\ x_{\lambda a} = \lambda x_a \quad \forall a,b \in A,\ \lambda \in \mathbb{C},\ |\lambda| \leq 1 \}$$

$\subset X$ closed in $X$

(since defined by equalities of continuous functions)

$\Rightarrow \operatorname{ran} T$ is compact w.r.t. $\tau_X$

We identify now $\Sigma$ with $\operatorname{ran} T \subset X$, this gives then topology $\tau$ on $\Sigma$

(via $\mathcal{U} \in \tau \Leftrightarrow \mathcal{U} = T^{-1}(V)$ for $V \in \tau_X$)

$\Rightarrow \tau$ compact

Furthermore,

$\hat{a} : \Sigma \to \mathbb{C}$, $\varphi \mapsto \varphi(a)$ corresponds

to $T(\Sigma) \to \mathbb{C}$ $\quad (\varphi(b))_{b\in A} \mapsto \varphi(a)$

thus: $\hat{a}$ corresponds to the projection $\pi_a$ in the product space and is thus continuous

14.4. Def: Let $A$ be a commutative Banach algebra.

1) The set $\Sigma$ of all complex homomorphisms on $A$, equipped with the canonical topology from 14.2., is the maximal ideal space or spectrum of $A$.

2) The map

$$\wedge : A \to C(\Sigma)$$
$$a \mapsto \hat{a}$$

where $\hat{a}(\varphi) = \varphi(a)$

is the Gelfand representation of $A$; and $\hat{a}$ is the Gelfand transform of $a$.

14.5. Theorem: Let $A$ be a commutative Banach algebra. Then the Gelfand representation is an algebra homomorphism and we have: i) $\hat{a}(\Sigma) = \sigma(a)$ $\forall a \in A$

ii) $\|\hat{a}\| = r(a) \leq \|a\|$ $\forall a \in A$

iii) $\varphi_1, \varphi_2 \in \Sigma$ with $\varphi_1 \neq \varphi_2$

$$\Rightarrow \exists a \in A : \hat{a}(\varphi_1) \neq \hat{a}(\varphi_2)$$

Proof: Homomorphismus clea, since

$$(\lambda a+\mu b)^{\wedge}(\varphi)=\varphi(\lambda a+\mu b)$$
$$=\lambda\,\varphi(a)+\mu\,\varphi(b)$$
$$=\lambda\,\hat{a}(\varphi)+\mu\,\hat{b}(\varphi)$$

$$\widehat{ab}(\varphi)=\varphi(ab)=\varphi(a)\varphi(b)=\hat{a}(\varphi)\,\hat{b}(\varphi)$$

$$\hat{1}(\varphi)=\varphi(1)=1$$

i) $\hat{a}(\Sigma)=\{\underbrace{\hat{a}(\varphi)}_{\varphi(a)}\mid\varphi\in\Sigma\}\underset{13.7(v)}{=}\sigma(a)$

ii) $\|\hat{a}\|=\sup\limits_{\varphi\in\Sigma}|\hat{a}(\varphi)|$

$$=\sup_{\varphi\in\Sigma}|\varphi(a)|=\sup_{\lambda\in\sigma(a)}\lambda=r(a)$$
$$\underset{12.9.}{\leq}\|a\|$$

iii) Assume $\underbrace{\hat{a}(\varphi_1)}_{\varphi_1(a)}=\underbrace{\hat{a}(\varphi_2)}_{\varphi_2(a)}\quad\forall a\in A$

$\Rightarrow\varphi_1=\varphi_2$

## 14.6 Theorem (Gelfand - Naimark):

Let $A$ be a commutative $C^*$-algebra. Then the Gelfand representation $\hat{}: A \to C(\Sigma)$ is an isometric $*$-isomorphism.

Proof: each $a \in A$ is normal

$$\overset{12.13}{\Longrightarrow} \quad r(a) = \|a\| \quad \forall a \in A$$

$$\Longrightarrow \quad \|\hat{a}\| = r(a) = \|a\| \quad \forall a \in A$$

i.e., $\hat{}$ is isometric

$\Rightarrow \hat{A}$ is a closed subalgebra of $C(\Sigma)$

14.5(iii) $\Longrightarrow$ $\hat{A}$ separates points of $\Sigma$

$1 = \hat{1} \in \hat{A} \Rightarrow \hat{A}$ unital subalgebra

$$\widehat{a^*}(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{\hat{a}(\varphi)}$$

$$\Longrightarrow \widehat{a^*} = \overline{\hat{a}}$$

thus: $\hat{a} \in \hat{A} \Longrightarrow \overline{\hat{a}} \in \hat{A}$

$$\overset{\text{Stone-}}{\underset{\text{Weierstraß}}{\Longrightarrow}} \quad \hat{A} = C(\Sigma)$$

<u>14.7. Proposition:</u> Let $A$ be a $C^*$-algebra and $a \in A$ <u>normal</u>, i.e., $aa^* = a^*a$. Put

$$C^*(a) := \overline{\{\text{polynomials in } a \text{ and } a^*\}}^{\|.\|},$$

the smallest $C^*$-algebra that contains $a$ (and thus also $a^*$). Then $C^*(a)$ is commutative, thus

$$C^*(a) \hat{=} C(\Sigma)$$

where $\Sigma = \Sigma_{C^*(a)}$ is the spectrum of $C^*(a)$. Furthermore,

$$\Sigma_{C^*(a)} \hat{=} \sigma(a) \quad \text{as topological spaces}$$

via the homeomorphism

$$\hat{a} : \Sigma_{C^*(a)} \to \sigma(a)$$

<u>Proof:</u> $\hat{a} \in C(\Sigma)$, i.e.,

$$\hat{a} : \Sigma \to \mathbb{C} \quad \text{continuous}$$

$$\operatorname{ran} \hat{a} = \{\hat{a}(\varphi) \mid \varphi \in \Sigma\} = \{\lambda \mid \lambda \in \sigma(a)\}$$

$$\Rightarrow \hat{a} : \Sigma \to \sigma(a) \quad \text{surjective}$$

Assume $\hat{a}(\varphi_1) = \hat{a}(\varphi_2)$, i.e.

$$\varphi_1(a) = \varphi_2(a)$$

$$\overset{13.8.}{\Longrightarrow} \varphi_1(a^*) = \overline{\varphi_1(a)} = \overline{\varphi_2(a)} = \varphi_2(a^*)$$

$$\Rightarrow \varphi_1(p(a,a^*)) = \varphi_2(p(a,a^*))$$

$\forall$ polynomials $p(\cdot,\cdot)$ in two commuting variables

$$\left.\begin{array}{l}\text{such polynomials } p(a,a^*) \\ \text{are dense in } C^*(a) \\ \varphi_1, \varphi_2 \text{ continuous} \\ (\text{as } \|\varphi_1\| = 1 = \|\varphi_2\|)\end{array}\right\} \Rightarrow \begin{array}{l}\varphi_1(x) = \varphi_2(x) \\ \forall x \in C^*(a)\end{array}$$

thus: $\varphi_1 = \varphi_2$

$\Rightarrow \hat{a}$ injective

$\Rightarrow \hat{a} : \Sigma_1 \to \sigma(a)$ bijective

and so: $\hat{a}$ is continuous bijection between compact spaces and thus automatically a homeomorphism (i.e. $\hat{a}^{-1}$ is also continuous) $\square$

## 14.9 Theorem (continuous functional calculus for normal elements)

Let $A$ be a $C^*$-algebra and $a \in A$ normal.

1) There exists exactly one $*$-homomorphism

$$\begin{aligned} \Phi : C(\sigma(a)) &\to A \\ f &\mapsto \Phi(f) \end{aligned}$$

s.th. $\Phi(z) = a$, where

$$\begin{aligned} z : \sigma(a) &\to \mathbb{C} \\ \lambda &\mapsto \lambda \end{aligned}$$

We will also denote $\Phi(f)$ by $f(a)$.

2) $\Phi$ is an isometry, i.e.,

$$\|f(a)\| = \|f\|_{C(\sigma(a))} = \sup_{\lambda \in \sigma(a)} |f(\lambda)|$$

and

$$\operatorname{ran} \Phi = C^*(a)$$

is the smallest $C^*$-algebra that contains $a$. [Note: $C^*(a)$ is commutative.]

Proof: i) Uniqueness

Let $\Phi$ be with properties as in (1)

$\Rightarrow$ $\Phi(p)$ is determined for any polynomial $p$ in $z$ and $\bar{z}$

$$p(z) = \sum \alpha_{nm} z^n \bar{z}^m \Rightarrow \Phi(p) = p(a)$$
$$= \sum \alpha_{nm} a^n a^{*m}$$

Since (Exercise!)

$p_n \longrightarrow f$ in $C(\sigma(a))$ $\Longrightarrow$ $\Phi(p_n) \longrightarrow \Phi(f)$ in $A$

and, by Stone–Weierstraß, polynomials $p(z,\bar{z})$ are dense in $C(\sigma(a))$

$\Longrightarrow$ $\Phi$ is determined on all of $C(\sigma(a))$!

ii) Existence

We have the following identifications

$$\wedge : C^*(a) \longrightarrow C(\Sigma_{C^*(a)}) \qquad \text{Gelfand isomorph.}$$
$$y \longmapsto \hat{y}$$

and via

$$\hat{a} : \Sigma_{C^*(a)} \longrightarrow \sigma(a)$$

also

$$\mathcal{C}(\sigma(a)) \longrightarrow \mathcal{C}(\Sigma_{\mathcal{C}^*(a)})$$
$$f \longmapsto f \circ \hat{a}$$

Thus

$$\mathcal{C}^*(a) \xrightarrow{\wedge} \mathcal{C}(\Sigma) \underset{\circ \hat{a}}{\overset{\circ \hat{a}^{-1}}{\rightleftarrows}} \mathcal{C}(\sigma(a))$$

and $\Phi(f)$ is uniquely determined by

$$\widehat{\Phi(f)} = f \circ \hat{a}$$

iii) Properties of $\Phi$

∘ isometry

$$\|\Phi(f)\|_A = \|\widehat{\Phi(f)}\|_{\mathcal{C}(\Sigma)}$$
$$= \|f \circ \hat{a}\|_{\mathcal{C}(\Sigma)}$$
$$= \sup_{\varphi \in \Sigma} |f(\hat{a}(\varphi))|$$
$$= \sup_{\lambda \in \sigma(a)} |f(\lambda)|$$
$$= \|f\|_{\mathcal{C}(\sigma(a))}$$

- $\Phi(z) = a$

  since: $\widehat{\Phi(z)} = z \circ \hat{a} = \hat{a}$

- $\bar{f}(a) = f(a)^*$, since

  $\widehat{f(a)} = f \circ \hat{a}$

  $\Rightarrow \widehat{f(a)^*} = \overline{\widehat{f(a)}} = \bar{f} \circ \hat{a}$

- homomorphism

$$\widehat{(f+g)(a)} = (f+g) \circ \hat{a}$$

$$= f \circ \hat{a} + g \circ \hat{a}$$

$$= \widehat{f(a)} + \widehat{g(a)}$$

$$= \widehat{f(a) + g(a)}$$

$\Rightarrow (f+g)(a) = f(a) + g(a)$

$(f \cdot g)(a) = f(a) \cdot g(a)$ similar

$\square$