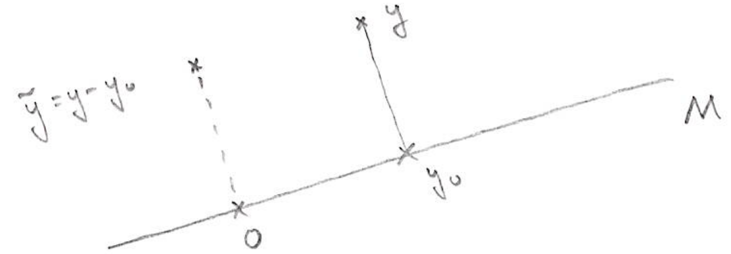
8. Geometry of Banach spaces

8.1. Lemma: Let X be a Banach space and $M \subsetneq X$ a linear subspace. Then, for each $\varepsilon > 0$ there is an $x \in X$ s.th. $\|x\| = 1$ and $\operatorname{dist}(x, M) \geq 1 - \varepsilon$.

Proof: Put $\delta(y) := \operatorname{dist}(y, M) \quad (y \in X)$



Consider arbitrary $y \in X \setminus M$

$\Rightarrow \exists\, y_0 \in M$ s.th. $\delta(y) \leq \|y_0 - y\| \leq (1+\varepsilon)\,\delta(y)$

Put $\tilde{y} := y - y_0 \neq 0$

$$\Rightarrow \delta(\tilde{y}) = \inf_{z \in M} \|\tilde{y} - z\| = \inf_{z \in M} \|\underbrace{y - y_0 - z}_{y - \tilde{z}\ (\tilde{z} \in M)}\| = \delta(y)$$

$$\Rightarrow (1+\varepsilon)\,\delta(\tilde{y}) \geq \|y_0 - y\| = \|\tilde{y}\|$$

Put now $x := \dfrac{\tilde{y}}{\|\tilde{y}\|} \quad \Rightarrow \quad \|x\| = 1$

and
$$\|z - x\| = \Big\|z - \frac{\tilde{y}}{\|\tilde{y}\|}\Big\| = \frac{1}{\|\tilde{y}\|} \underbrace{\|\underbrace{z\,\|\tilde{y}\|}_{\in M} - \tilde{y}\|}_{\substack{\geq\, \delta(\tilde{y}) \\ \geq\, \frac{\|\tilde{y}\|}{1+\varepsilon}}} \geq \frac{1}{1+\varepsilon} > 1 - \varepsilon \qquad \forall z \in M$$

$\Rightarrow \operatorname{dist}(x, M) \geq 1 - \varepsilon$ $\square$

8.2. Proposition: Let X be a Banach space and

$$B_1(X) := \{ x \in X \mid \|x\| \leq 1 \}$$

the closed unit ball in X.

Then the following statements are equivalent:

a) X is finite-dimensional vector space

b) $B_1(X)$ is compact

Proof: (a) $\Rightarrow$ (b) $\dim X < \infty \Rightarrow$ all norms are equivalent (without proof)

$\Rightarrow X \mathrel{\hat{=}} (\mathbb{C}^n, |\cdot|)$

$B_1(X)$ closed and bounded $\xRightarrow[\text{Borel}]{\text{Heine-}}$ $B_1(X)$ compact

(b) $\Rightarrow$ (a) Choose inductively a sequence $(x_n)_{n \in \mathbb{N}}$ in X s.th.

$\|x_n\| = 1 \quad \forall n$

$\operatorname{dist}(x_n, \operatorname{span}(x_1, \dots, x_{n-1})) \geq \frac{1}{2}$ (possible by 8.1)

$\Rightarrow (x_n) \subset B_1(X)$, but without convergent subsequence,

because $\|x_n - x_m\| \geq \frac{1}{2} \quad \forall n \neq m$

$\square$

8.3. Def.: 1) Let X be a Banach space and $M, N \subset X$ closed linear subspaces (i.e. M, N Banach spaces). M and N are <u>complementary</u> if

$$M \cap N = \{0\} \quad \text{and} \quad M + N = X$$

$$M + N = \{x + y \mid x \in M, y \in N\}$$

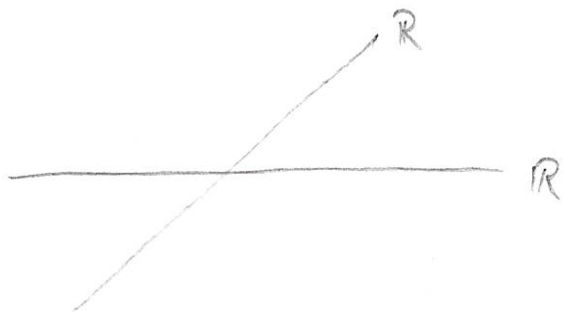
We write in this case: $X = M \oplus N$

2) A closed linear subspace $M \subset X$ is <u>complemented</u>, if there is a closed linear subspace $N \subset X$ s.th.:

$$X = M \oplus N.$$

8.4. Remarks: 1) complementary space in general not unique;

e.g.: $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$



2) $X = \mathcal{H}$ Hilbert space $\Rightarrow$ each closed linear subspace $\mathcal{H}_0$ is complemented: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$

3) in Banach spaces this is not true;

e.g.: c_0 is not complemented in ℓ_∞ (Phillips 1940)

4) We even have (Lindenstrauss + Tzafriri 1971):

$$\left.\begin{array}{l} X \text{ Banach space} \\ \text{each subspace complemented} \end{array}\right\} \Rightarrow X \text{ Hilbert space}$$

8.5. Prop.: Let V be a finite-dimensional vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ two norms on V. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, i.e. $\exists\, C_1 > 0$ s.th.

$$\frac{1}{C_1}\|x\|_1 \le \|x\|_2 \le C_1\|x\|_1 \qquad \forall x \in V$$

Proof: it suffices to take special norm for $\|\cdot\|_1$:

Let $e_1, \dots, e_n$ basis of V, then put

$$\Big\|\sum_{i=1}^{n} \alpha_i e_i\Big\|_1 := \sum_{i=1}^{n} |\alpha_i|$$

Consider $x = \sum \alpha_i e_i$

$$\Rightarrow \|x\|_2 = \Big\|\sum \alpha_i e_i\Big\|_2 \le \sum |\alpha_i|\,\|e_i\|_2 \le C_1 \cdot \sum |\alpha_i| = C_1 \cdot \|x\|_1$$

where $C_1 := \max\limits_{1\le i\le n} \|e_i\|_2$

i.e: $\|\cdot\|_2 : (V, \|\cdot\|_1) \to \mathbb{R}_+$ continuous

i.e. $x_n \xrightarrow{\|\cdot\|_1} x \Rightarrow \|x_n\|_2 \to \|x\|_2$

(this follows from $\|x_n - x\|_1 \to 0 \Rightarrow \|x_n - x\|_2 \to 0$)

Put $S := \{x \in V \mid \|x\|_1 = 1\}$ compact

$\Rightarrow$ inf of $\|\cdot\|_2$ is achieved on S, i.e.

$$\inf_{x\in S} \|x\|_2 = \min_{x \in S} \|x\|_2 =: \delta > 0$$

$$\Rightarrow \|x\|_2 = \|\alpha x_0\|_2 = \alpha \|x_0\|_2 \ge \alpha \cdot \delta = \delta \|x\|_1 \qquad \forall x \in V$$

$x = \alpha x_0$

$\|x_0\|_1 = 1$

$\square$