6. The Hahn-Banach Theorem

6.1. Def.: Let V be a vector space over $\mathbb{F}$.

1) A sublinear functional is a function

$q: V \to \mathbb{R}$ such that

i) $q(x+y) \leq q(x) + q(y)$ $\quad \forall x, y \in V$

ii) $q(\alpha x) = \alpha q(x)$ $\quad \forall x \in V, \alpha \geq 0$

2) A seminorm is a function

$p: V \to [0, \infty)$ s.th.

i) $p(x+y) \leq p(x) + p(y)$ $\quad \forall x, y \in V$

ii) $p(\alpha x) = |\alpha| p(x)$ $\quad \forall x \in V, \alpha \in \mathbb{F}$

6.2. Remark: A seminorm is a sublinea functional, but not conversely.

6.3. Theorem (Hahn-Banach, real version):

Let V be a vector space over $\mathbb{R}$ and q a sublinear functional on V. Let M be a linear subspace of V and $\ell: M \to \mathbb{R}$ a linear functional s.th. $\ell(x) \leq q(x)$ for all $x \in M$. Then there is a linear functional $L: V \to \mathbb{R}$ s.th. $L|M = \ell$ and

$L(x) \leq q(x)$ for all $x \in V$.

Proof: (1) Extension to one dimension more:

Fix $x_0 \in V \setminus M$ and consider

$$V_0 := \operatorname{span}(V, x_0) = \{ y + t x_0 \mid y \in M, t \in \mathbb{R} \}$$

to show: $\exists\, L_0 : V_0 \to \mathbb{R}$ s.th $L_0|M = \ell$ and

$$L_0(x) \leq q(x) \quad \forall x \in V_0$$

note: representation $x = y + t x_0$ for $x \in V_0$ is unique,

because $y + t x_0 = \tilde{y} + \tilde{t} x_0 \implies x_0 = \frac{y - \tilde{y}}{\tilde{t} - t} \in M$

Thus we can define an L_0 by choosing $\alpha \in \mathbb{R}$ and putting

$$L_0(y + t x_0) = \ell(y) + t\, L_0(x_0) = \ell(y) + t \cdot \alpha$$

Problem: Choose α such that $L_0(x) \leq q(x)$, i.e.

i) $L_0(y + t x_0) \leq q(y + t x_0)$

ii) $L_0(y - t x_0) \leq q(y - t x_0)$

$\forall y \in M, t \geq 0$

(i) $\iff \ell(y) + t\alpha \leq q(y + t x_0)$

$$\iff \alpha \leq \underbrace{\frac{1}{t} q(y + t x_0)}_{q(y/t + x_0)} - \ell\left(\frac{y}{t}\right)$$

$$\iff \alpha \leq q(y_1 + x_0) - \ell(y_1) \quad \forall y_1 \in M$$

(ii) $\Leftrightarrow \ell(y) - t\alpha \leq q(y - t x_0)$

$\Leftrightarrow \alpha \geq \ell(y/t) - \underbrace{\tfrac{1}{t} q(y - t x_0)}_{q(y/t - x_0)}$

$\Leftrightarrow \alpha \geq \ell(y_2) - q(y_2 - x_0) \qquad \forall y_2 \in M$

thus to show: $\exists \alpha$ s.th.

$\ell(y_2) - q(y_2 - x_0) \leq \alpha \leq q(y_1 + x_0) - \ell(y_1) \quad \forall y_1, y_2 \in M$

i.e.: $\ell(y_1 + y_2) \leq q(y_1 + x_0) + q(y_2 - x_0)$ — " —

But this follows from:

$$\ell(y_1 + y_2) \leq q(y_1 + y_2) = q((y_1 + x_0) + (y_2 - x_0))$$
$$\leq q(y_1 + x_0) + q(y_2 - x_0)$$

since q sublinear %.

② "induction" from M to V (à la Zorn):

$\mathcal{S} := \{ (V_0, L_0) \mid V_0$ linear subspace of V s.th. $M \subseteq V_0$, $L_0 : V_0 \to \mathbb{R}$ linear functional s.th. $L_0|M = \ell$, $L_0 \leq q$ on $V_0 \}$

For $(V_0, L_0), (V_1, L_1) \in \mathcal{S}$ we define

$$(V_0, L_0) \precsim (V_1, L_0) :\Longleftrightarrow V_0 \subseteq V_1, \; L_1|V_0 = L_0$$

$\Rightarrow (\mathcal{S}, \precsim)$ partially ordered set

Let $\mathcal{C} = \{(V_i, L_i) \mid i \in I\}$ be a chain in $\mathcal{S}$

(i.e. every two elements can be compared)

$\Rightarrow \exists$ upper bound (N, L) of $\mathcal{C}$, namely

$N = \bigcup_{i \in I} V_i$ (linear, because $\mathcal{C}$ chain)

$L(x) = L_i(x)$ if $x \in V_i$

Zorn's Lemma $\Rightarrow \exists$ maximal element (Z, L) of $\mathcal{C}^{\mathcal{S}}$

$\Rightarrow Z = V$, because otherwise step ① would give an extension

$\Rightarrow L : Z = V \to \mathbb{R}$ is wanted extension $\square$

6.4. Theorem (Hahn-Banach, complex version):

Let V be a vector space over $\mathbb{F}$ and $p: V \to [0, \infty)$ a seminorm. Let M be a linear subspace of V and $\ell: M \to \mathbb{F}$ a linear functional s.th. $|\ell(x)| \leq p(x)$ for all $x \in M$. Then there is a linear functional $L: V \to \mathbb{F}$ s.th. $L|M = \ell$ and $|L(x)| \leq p(x)$ for all $x \in V$.

Proof: ① $\mathbb{F} = \mathbb{R}$

$$\ell(x) \le |\ell(x)| \le p(x) \overset{6.3.}{\Longrightarrow} \exists\, L : V \to \mathbb{R} \text{ s.th.}$$

$$L|M = \ell \quad \text{and}$$

$$L(x) \le p(x) \quad \forall x \in V$$

but also: $-L(x) = L(-x) \le p(-x) = p(x)$ (p seminorm)

$\Rightarrow |L(x)| \le p(x) \quad \forall x \in V$

② $\mathbb{F} = \mathbb{C}$

Put $\ell_1 := \operatorname{Re} \ell$, i.e. $\ell_1 : M \to \mathbb{R}$ $\mathbb{R}$-linear s.th.

$$|\ell_1(x)| \le |\ell(x)| \le p(x)$$

$\overset{\text{case ①}}{\Longrightarrow} \exists\, L_1 : V \to \mathbb{R}$ $\mathbb{R}$-linear s.th.

$$L_1|M = \ell_1 \quad \text{and} \quad |L_1(x)| \le p(x) \quad \forall x \in V$$

note: ℓ can be recovered from ℓ_1 by:

$$\ell_1(x) = \operatorname{Re} \ell(x)$$

$\Rightarrow \ell_1(ix) = \operatorname{Re} \ell(ix) = \operatorname{Re} i\,\ell(x) = -\operatorname{Im} \ell(x)$

$\Rightarrow \ell(x) = \ell_1(x) - i\,\ell_1(ix)$

Thus we define L by

$$L(x) := L_1(x) - i\,L_1(ix)$$

$$\left.\begin{aligned} &\Rightarrow L \ \mathbb{R}\text{-linear} \\ &L(ix) = L_1(ix) - iL_1(-x) \\ &\quad = iL_1(x) + L_1(ix) \\ &\quad = iL(x) \end{aligned}\right\} \Rightarrow L \ \mathbb{C}\text{-linear}$$

$L|M = \ell$ clear by construction

to show: $|L(x)| \leq p(x) \quad \forall x \in V$

$x \in V \leadsto$ Choose μ s.th $|\mu| = 1$ and $|L(x)| = \mu L(x)$

$$\Rightarrow |L(x)| = \mu L(x) = L(\mu x) \underset{\substack{\uparrow \\ \text{since real}}}{=} \operatorname{Re} L(\mu x) = L_1(\mu x)$$
$$\leq p(\mu x)$$
$$= p(x)$$

□

<u>6.5 Corollary:</u> Let X be a normed space, $M \subset X$ a linear subspace, and $\ell : M \to \mathbb{F}$ a bounded linear functional. Then there is an $L \in X^*$ s.th.

$L|M = \ell$ and $\|L\| = \|\ell\|$

<u>Proof:</u> Use 6.4. with $p(x) = \|\ell\| \cdot \|x\|$

$$|\ell(x)| \leq \underbrace{\|\ell\| \cdot \|x\|}_{p(x)} \Rightarrow |L(x)| \leq \|\ell\| \cdot \|x\|$$
$$\Rightarrow \|L\| \leq \|\ell\|$$

□

<u>6.6. Corollary:</u> Let X be a normed space and $x \in X$. Then

$$\|x\| = \sup_{\substack{L \in X^* \\ \|L\| = 1}} |L(x)|$$

Moreover, the supremum is attained, i.e. $\forall x \in X$ $\exists L \in X^*$ s.th. $\|L\| = 1$ and $L(x) = \|x\|$.

<u>Proof:</u> $\sup |L(x)| \leq \|x\|$ clear, since $|L(x)| \leq \|L\| \cdot \|x\|$

"$=$": Let $M := \{\alpha x \mid \alpha \in \mathbb{F}\}$ one-dimensional subspace of X

$\ell : M \to \mathbb{F}$ defined by $\ell(\alpha x) := \alpha \|x\|$

$\Rightarrow \ell \in M^*$, $\|\ell\| = 1$

$\overset{6.5}{\Longrightarrow}$ $\exists L \in X^*$ s.th. $L|M = \ell$ and $\|L\| = \|\ell\| = 1$

$$\Downarrow$$

$$L(x) = \ell(x) = \|x\|$$

$\square$

<u>6.7. Remark:</u> The "duality"

$$\|L\| = \sup_{\substack{x \in X \\ \|x\| = 1}} |L(x)| \quad \longleftrightarrow \quad \|x\| = \sup_{\substack{L \in X^* \\ \|L\| = 1}} |L(x)|$$

suggests to write

$L(x) = \langle L, x \rangle$ for $x \in X$, $L \in X^*$

$\langle \cdot, \cdot \rangle$ pairing between X and X^*

(note: X Hsp $\Rightarrow$ $X \hat{=} X^*$, $\langle \cdot, \cdot \rangle \hat{=}$ inner product)

Quite often one also writes x^* for elements from X^*.

<u>Question:</u> What is X^{**}

Hilbert spaces, reflexive Banachspaces : $X^{**} = X$

in general ?

<u>6.8 Remark:</u> Consider $x \in X$, define $\hat{x}$ by

$\hat{x}(x^*) = x^*(x)$, then

- $\hat{x}$ linear
- $$\|\hat{x}\| = \sup_{\substack{x^* \in X \\ \|x^*\| = 1}} |\hat{x}(x^*)| = \sup_{\substack{x^* \in X \\ \|x^*\| = 1}} |x^*(x)| \overset{6.6}{=} \|x\|$$

thus: $\hat{x} \in X^{**}$

Hence: $\begin{array}{l} X \to X^{**} \\ x \mapsto \hat{x} \end{array}$ is isometric embedding

<u>6.9. Def:</u> If the canonical isometric embedding

$\hat{\ } : X \to X^{**}$, $x \mapsto \hat{x}$, where $\hat{x}(x^*) = x^*(x)$ $\forall x^* \in X^*$

is a bijection then X is called <u>reflexive</u>.