6. The Hahn-Banach Theorem

6.1. Def.: Let V be a vector space over $\mathbb{F}$.

1) A *sublinear functional* is a function

$q : V \to \mathbb{R}$ such that

i) $q(x+y) \leq q(x) + q(y) \qquad \forall x, y \in V$

ii) $q(\alpha x) = \alpha\, q(x) \qquad \forall x \in V, \alpha \geq 0$

2) A *seminorm* is a function

$p : V \to [0, \infty)$ s.th.

i) $p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in V$

ii) $p(\alpha x) = |\alpha|\, p(x) \qquad \forall x \in V, \alpha \in \mathbb{F}$

6.2. Remark: A seminorm is a sublinea functional, but not conversely.

6.3. Theorem (Hahn-Banach, real version):
Let V be a vector space over $\mathbb{R}$ and q a sublinear functional on V. Let M be a linear subspace of V and $\ell : M \to \mathbb{R}$ a linear functional s.th. $\ell(x) \leq q(x)$ for all $x \in M$. Then there is a linear functional $L : V \to \mathbb{R}$ s.th. $L|M = \ell$ and

$L(x) \leq q(x)$ for all $x \in V$.