1. Hilbert spaces

$\mathbb{F}$ field, usually $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

1.1. Def.: A vector space over $\mathbb{F}$ is a set V with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ such that

(i) $(V, +)$ is abelian group

(ii) $\lambda(x+y) = \lambda x + \lambda y \quad \forall \lambda \in \mathbb{F}, x, y \in V$

(iii) $(\lambda + \mu) x = \lambda x + \mu x \quad \forall \lambda, \mu \in \mathbb{F}, x \in V$

(iv) $(\lambda \mu) x = \lambda(\mu x) \quad \forall \lambda, \mu \in \mathbb{F}, x \in V$

(v) $1 . x = x \quad \forall x \in V$

1.2. Def.: 1) If V is a vector space over $\mathbb{F}$, an inner product on V is a function

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$$

with the following properties:

(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle \quad \forall \lambda, \mu \in \mathbb{F}$, $\forall x, y, z \in V$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in V$

(iii) $\langle x, x \rangle \geq 0 \quad \forall x \in V$

(iv) $\langle x, x \rangle = 0 \iff x = 0$

2) A pre-Hilbert space is a vector space together with an inner product.

1.3. Examples: 1) $V = \mathbb{F}^n$

$$x = (\alpha_1, \dots, \alpha_n) \qquad \alpha_1, \dots, \alpha_n \in \mathbb{F}$$

$$\hat{=}\ \alpha_1 \vec{e_1} + \dots + \alpha_n \vec{e_n}$$

$$\langle (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \rangle = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

remind: $\left| \sum_{i=1}^{n} \alpha_i \overline{\beta_i} \right| \leq \left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{n} |\beta_i|^2 \right)^{1/2}$

Cauchy - Schwarz inequality

2) $V = L^2(0,1) = \{ f \mid \int_0^1 |f(t)|^2 \, dt < \infty \}$

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, dt$$

note: $f, g \in L^2 \Rightarrow$

$$\left| \int_0^1 f(t) \bar{g}(t) \, dt \right| \leq \left(\int_0^1 |f(t)|^2 \, dt \right)^{1/2} \cdot \left(\int_0^1 |g(t)|^2 \, dt \right)^{1/2}$$

Hölder's inequality

1.4 Theorem (Cauchy-Schwarz Inequality)

1) If $\langle \cdot , \cdot \rangle$ is an inner product on V, then

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle \cdot \langle y,y\rangle \qquad \forall x,y \in V$$

2) We have "=" if and only if

x and y are linearly dependent

Proof: 1) We have $\forall \lambda \in \mathbb{F}$:

$$0 \leq \langle \lambda x + y , \lambda x + y \rangle$$

$$= \lambda \bar{\lambda} \langle x,x\rangle + \lambda \langle x,y\rangle + \bar{\lambda} \langle y,x\rangle + \langle y,y\rangle .$$

If $\langle x,x\rangle = 0 \Rightarrow \langle x,y\rangle = 0 \quad \forall y \in V$

(i.e. $x = 0$) $\Rightarrow$ Ineq. true

Otherwise put $\lambda = -\dfrac{\langle y,x\rangle}{\langle x,x\rangle}$

$$\Rightarrow 0 \leq \underbrace{\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle} - 2\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}}_{-\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}} + \langle y,y\rangle$$

$$\Rightarrow \langle y,y\rangle \geq \frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}$$

2) $$\text{"="} \Leftrightarrow \left\{ \begin{array}{l} x = 0 \text{ or} \\ \langle \lambda x+y, \lambda x+y\rangle = 0 \Leftrightarrow \lambda x + y = 0 \end{array} \right\} \Leftrightarrow \text{lin. dep.}$$

$\square$

We will put

$$\|x\| := \sqrt{\langle x, x\rangle} \qquad \text{"length" of } x$$

$V = \mathbb{F}^n$: $\quad x = (\alpha_1, \dots, \alpha_n)$

$$\Rightarrow \|x\| = \sqrt{|\alpha_1|^2 + \dots + |\alpha_n|^2}$$

<u>1.5. Def.</u>: 1) If V is a vector space over $\mathbb{F}$, a <u>norm</u> on V is a function

$\|\cdot\| : V \to \mathbb{R}_+$ such that

(i) $\|\lambda x\| = |\lambda| \, \|x\| \qquad \forall \lambda \in \mathbb{F}, x \in V$

(ii) $\|x+y\| \leq \|x\| + \|y\| \qquad \forall x, y \in V$

("triangle inequality")

(iii) $\|x\| = 0 \iff x = 0$

2) A <u>normed space</u> is a vector space together with a norm.

<u>1.6. Proposition</u>: If V is a pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then

$$\|x\| := \sqrt{\langle x, x\rangle} \qquad (x \in V)$$

defines a norm on V.

Thus: a pre-Hilbert space is a normed space

Proof: - $\|x\| \geq 0$ ✓

- $\|\lambda x\| = \sqrt{<\lambda x, \lambda x>} = \sqrt{\lambda \bar{\lambda} <x, x>} = |\lambda| \, \|x\|$
- $\|x\| = 0 \Leftrightarrow <x, x> = 0 \Leftrightarrow x = 0$
- $$\begin{aligned} \|x+y\|^2 &= <x+y, x+y> \\ &= <x,x> + <x,y> + <y,x> + <y,y> \\ &= \|x\|^2 + 2 \underbrace{\operatorname{Re} <x,y>}_{\substack{\leq |<x,y>| \\ \leq \|x\| \cdot \|y\| \ (\text{C-S})}} + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$ □

1.7. Def.: 1) A normed space V is complete, if every Cauchy sequence converges, i.e.

$$\left.\begin{array}{l} (x_n) \text{ CS}: \ x_n \in V \text{ and} \\ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that} \\ \|x_n - x_m\| \leq \varepsilon \quad \forall n, m \geq N \end{array}\right\} \Rightarrow \begin{array}{l} \exists x \in V \text{ such that } x_n \to x \\ (\text{i.e.: } \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: \\ \quad \|x_n - x\| \leq \varepsilon \ \forall n \geq N) \\ (\text{i.e.: } \|x_n - x\| \to 0) \end{array}$$

2) A Hilbert space $\mathcal{H}$ is a pre-Hilbert space that is complete (with respect to the norm $\|x\| = \sqrt{<x,x>}$).

3) A Banach space X is a normed space that is complete.

thus: a Hilbert space is a Banach space

1.8. Examples: 1) $\mathbb{F}^n$, $L^2(0,1)$ are Hilbert spaces

2) $\ell_2 \cong \mathbb{F}^\infty$

$$\ell_2 := \{ (\alpha_i)_{i=1}^\infty \mid \alpha_i \in \mathbb{F}, \sum_{i=1}^\infty |\alpha_i|^2 < \infty \}$$

space of square-summable sequences

$$\langle (\alpha_i), (\beta_j) \rangle := \sum_{i=1}^\infty \alpha_i \overline{\beta_i}$$

ℓ_2 is Hilbert space

3) Each closed linear subspace of Hilbertspace (Banach space) is a Hilbert space (Banach space).

4) $C[a,b] := \{ f : [a,b] \to \mathbb{F} \mid f \text{ continuous} \}$

$$\|f\| := \sup_{t \in [a,b]} |f(t)|$$

is Banach space, but not Hilbert space

5) $C[a,b]$ is w.r.t. $\langle f, g \rangle := \int_a^b f(t) \overline{g(t)}\, dt$

pre-Hilbert space, but not complete.

$$\left(\overline{C[a,b]}^{\langle \cdot, \cdot \rangle} = L^2[a,b] \right)$$

1.9. Prop.: 1) Let X be a Banach space. Then the maps (for $\alpha \in \mathbb{F}$)

$x \mapsto \|x\|$, $(x,y) \mapsto x+y$, $x \mapsto \alpha x$ are continuous

2) Let $\mathcal{H}$ be a Hilbert space. Then, for fixed $y \in \mathcal{H}$, the maps $x \mapsto \langle x,y\rangle$ and $x \mapsto \langle y,x\rangle$ are continuous.

Proof: 1) $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$$\left.\begin{aligned} &\Rightarrow \|x\| - \|y\| \leq \|x-y\| \\ &\overset{x \leftrightarrow y}{\Rightarrow} \|y\| - \|x\| \leq \|x-y\| \end{aligned}\right\} \Rightarrow \big|\|x\| - \|y\|\big| \leq \|x-y\|$$

thus: $x_n \to x$ (i.e. $\|x_n - x\| \to 0$)

$\Rightarrow \big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| \to 0$

$\Rightarrow \|x_n\| \to \|x\|$

⊛

2) $x_n \to x \Rightarrow |\langle x_n,y\rangle - \langle x,y\rangle| =$

$$= |\langle x_n - x, y\rangle| \overset{CS}{\leq} \underbrace{\|x_n - x\|}_{\to 0} \cdot \|y\|$$

$\Rightarrow \langle x_n,y\rangle \to \langle x,y\rangle$

□

⊛ $x_n \to x$, $y_n \to y$

$\Rightarrow \|(x_n+y_n) - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$

$\Rightarrow x_n + y_n \to x+y$

1.10. Def.: Let V be a pre-Hilbert space.

1) $x, y \in V$ are <u>orthogonal</u> $(x \perp y)$, if

$$\langle x, y \rangle = 0$$

2) Subsets $A, B \subset V$ are <u>orthogonal</u> $(A \perp B)$,

if $\langle x, y \rangle = 0 \quad \forall x \in A,\ y \in B$

3) If $A \subset V$, then

$$A^{\perp} := \{ x \in V \mid x \perp A \}$$

is called <u>orthogonal complement</u> of A.

1.11. Theorem: Let V be a pre-Hilbert space.

Then we have:

1) Polarisation Identity

$$\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y \rangle + \|y\|^2$$

2) Parallelogram Law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

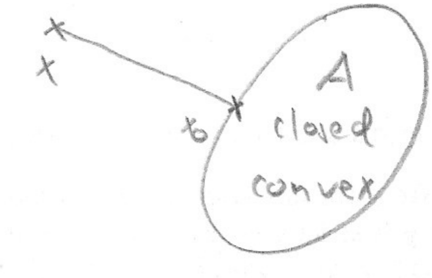
3) Pythagorean Theorem:

If $x_1, \dots, x_n \in V$ such that $x_i \perp x_j \ (i \neq j)$,

then

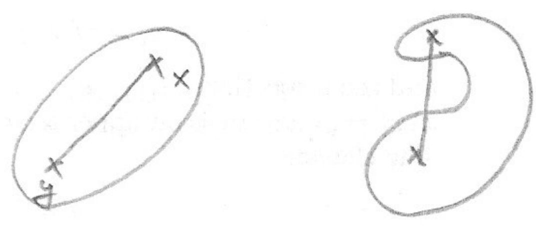
$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2$$

Proof: Exercise!



1.12. Def.: If V is a vector space over $\mathbb{F}$ and $A \subset V$, then A is <u>convex</u>, if

$$\left.\begin{array}{l} x, y \in A \\ 0 \le t \le 1 \end{array}\right\} \Rightarrow tx + (1-t)y \in A$$



convex not convex

1.13. Theorem: Let $A \neq \emptyset$ be a closed convex subset of a Hilbert space $\mathcal{H}$. Then, for each $x \in \mathcal{H}$, there is a unique point $x_0 \in A$ such that

$$\|x - x_0\| = \operatorname{dist}(x, A) := \inf_{y \in A} \|x - y\|$$

Proof: By def. of $\operatorname{dist}(x, A)$: $\exists\, y_n \in \overset{A}{\cancel{\mathcal{H}}}$ such that

$$\|x - y_n\| \to \operatorname{dist}(x, A)$$

want to show: y_n converges

suffices: $\{y_n\}$ Cauchy sequence

use Parallelogram Law for $x-y_n$ and $x-y_m$

$$\Rightarrow \|x-y_n + x-y_m\|^2 + \|x-y_n-(x-y_m)\|^2 =$$

$$= 2\left(\|x-y_n\|^2 + \|x-y_m\|^2\right)$$

$$\Rightarrow \|y_m-y_n\|^2 = 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right) - \underbrace{4\|x - \underbrace{\tfrac{y_n+y_m}{2}}_{\in A}\|^2}_{\geq \operatorname{dist}(x,A)^2}$$

$$\leq 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right) - 4 \operatorname{dist}(x,A)^2$$

$\leq \varepsilon$ for n, m sufficiently large

$\Rightarrow \{y_n\}$ CS

$\Rightarrow \exists\, x_0 \in \mathcal{H}$ s.th. $y_n \to x_0$ (i.e. $\|x_0 - y_n\| \to 0$)

A closed $\Rightarrow x_0 \in A$

$$\operatorname{dist}(x,A) \underset{x_0 \in A}{\leq} \|x-x_0\| = \|x-y_n+y_n-x_0\|$$

$$\leq \underbrace{\|x-y_n\|}_{\downarrow\ \operatorname{dist}(x,A)} + \underbrace{\|y_n-x_0\|}_{\downarrow\ 0}$$

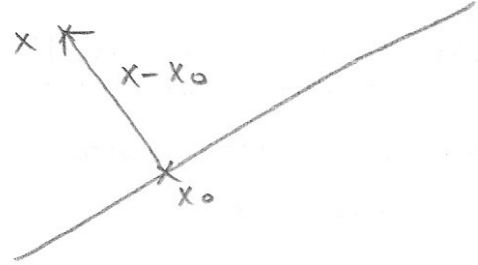
$\Rightarrow \|x-x_0\| = \operatorname{dist}(x,A)$

Uniqueness: Assume $\|x-x_1\| = \operatorname{dist}(x,A) = \|x-x_0\|$ $\square$

$\Rightarrow (y_n) = (x_0, x_1, x_0, x_1, \dots)$ is sequence as above $\Rightarrow$ (y_n) converges $\Rightarrow x_0 = x_1$

Consider now

$A = \mathcal{H}_0$ sub Hilbert space (closed linear subspace)



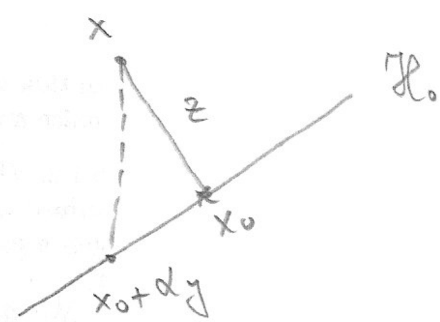
$A = \mathcal{H}_0$ (note: linear $\Rightarrow$ convex)

<u>1.14. Theorem</u>: Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed linear subspace. Then, for $x \in \mathcal{H}$ and $x_0 \in \mathcal{H}_0$, the following are equivalent:

(a) $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$

(b) $x - x_0 \perp \mathcal{H}_0$

<u>Proof</u>: (a) $\Rightarrow$ (b): Consider $y \in \mathcal{H}_0$



to show: $\langle x - x_0, y \rangle = 0$

suffices to restrict to $\|y\| = 1$

Put $z := x - x_0$, then $\forall \alpha \in \mathbb{F}$:

$$\|z\|^2 = \operatorname{dist}(x, \mathcal{H}_0)^2 \leq \| x - \underbrace{(x_0 + \alpha y)}_{\in \mathcal{H}_0} \|^2$$

$$= \|z - \alpha y\|^2$$

$$= \langle z - \alpha y, z - \alpha y \rangle$$

$$= \langle z, z \rangle - \alpha \langle y, z \rangle - \bar{\alpha} \langle z, y \rangle + |\alpha|^2 \underbrace{\langle y, y \rangle}_{1}$$

Choose $\alpha = \langle z, y \rangle$: $0 \leq -|\alpha|^2 \Rightarrow \underset{\langle z, y \rangle}{\underset{\shortparallel}{\alpha}} = 0$

b) $\Rightarrow$ a) $\quad x - x_0 \perp \mathcal{H}_0$

to show: $\|x - x_0\| \leq \|x - y\| \quad \forall y \in \mathcal{H}_0$

Consider $y \in \mathcal{H}_0$:

$$\|x-y\|^2 = \|\underbrace{(x - x_0)}_{\perp \mathcal{H}_0} + \underbrace{(x_0 - y)}_{\in \mathcal{H}_0}\|^2$$

$$\overset{\text{Pythagoras}}{=} \|x - x_0\|^2 + \|x_0 - y\|^2$$

$$\geq \|x - x_0\|^2$$

□

1.15. Def.: Let $\mathcal{H}_0, \mathcal{H}_1 \subseteq \mathcal{H}$ be closed linear subspaces of a Hilbertspace $\mathcal{H}$ such that $\mathcal{H}_0 \perp \mathcal{H}_1$. Then

$$\mathcal{H}_0 \oplus \mathcal{H}_1 := \mathcal{H}_0 + \mathcal{H}_1 = \{x + y \mid x \in \mathcal{H}_0, y \in \mathcal{H}_1\} \subseteq \mathcal{H}$$

is called (orthogonal) direct sum of $\mathcal{H}_0$ and $\mathcal{H}_1$.

1.16. Remark: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ means:

- each $x \in \mathcal{H}$ can be written as

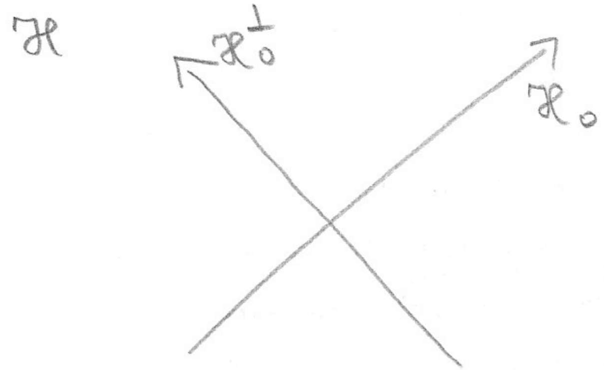
  $x = x_0 + x_1 \quad$ where $\quad x_0 \in \mathcal{H}_0, x_1 \in \mathcal{H}_1$

- this representation is unique

  $x = x_0 + x_1 = x_0' + x_1'$

  $$\Rightarrow \underbrace{x_0 - x_0'}_{\in \mathcal{H}_0} = \underbrace{x_1' - x_1}_{\in \mathcal{H}_1} \in \mathcal{H}_0 \cap \mathcal{H}_1 \overset{(*)}{=} \{0\}$$

(*): $\mathcal{H}_0 \perp \mathcal{H}_1$: $x \in \mathcal{H}_0 \cap \mathcal{H}_1$: $\langle \underset{\mathcal{H}_0}{x}, \underset{\mathcal{H}_1}{x} \rangle = 0 \Rightarrow x = 0$



1.17. Theorem: Let $\mathcal{H}_0 \subset \mathcal{H}$ be a closed linear subspace. Then we have:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

Proof: (*) $\mathcal{H}_0 \perp \mathcal{H}_0^\perp$ clear

remains to show: $\mathcal{H}_0 + \mathcal{H}_0^\perp = \mathcal{H}$

Consider $x \in \mathcal{H}$

$\overset{1.12}{\Longrightarrow}$ $\exists\, x_0 \in \mathcal{H}_0$ s.th. $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$

$\overset{1.14}{\Longrightarrow}$ $x - x_0 \perp \mathcal{H}_0$, i.e. $x - x_0 \in \mathcal{H}_0^\perp$

$$\Rightarrow \quad x = \underset{\mathcal{H}_0}{\underset{\Uparrow}{x_0}} + \underset{\mathcal{H}_0^\perp}{\underset{\Uparrow}{(x - x_0)}}$$

$\square$

(*) note: $A^\perp$ is always closed, for any $A \subset \mathcal{H}$; $\mathcal{H}_0^\perp$ is also linear subspace

1.18. Theorem: Let $\mathcal{H}_0 \subset \mathcal{H}$ be closed linear subspace.

For $x \in \mathcal{H}$, define $Px = x_0$ where $x_0 \in \mathcal{H}_0$ with

$\|x - x_0\| = \text{dist}(x, \mathcal{H}_0)$ $\quad (\Leftrightarrow x - x_0 \perp \mathcal{H}_0)$

Then

1) $P: \mathcal{H} \to \mathcal{H}$ is linear mapping

2) $\|Px\| \leq \|x\| \qquad \forall x \in \mathcal{H}$

3) $P^2 = P \qquad (P^2 x = P(Px))$

4) $\ker P = \mathcal{H}_0^\perp$, $\quad \operatorname{ran} P = \mathcal{H}_0$

($\ker P = \{x \in \mathcal{H} \mid Px = 0\}$ kernel

$\operatorname{ran} P = \{y \in \mathcal{H} \mid y = Px \text{ for some } x \in \mathcal{H}\}$ range)

Proof: Exercise!

1.19. Notation: P is called orthogonal projection

from $\mathcal{H}$ onto $\mathcal{H}_0$.

1.20. Def.: A <u>linear functional</u> is a map

$L : \mathcal{H} \to \mathbb{F}$ such that

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \qquad \forall \alpha, \beta \in \mathbb{F}, \; x, y \in \mathcal{H}$$

1.21. Remark: in finite dimensions: linear $\Rightarrow$ continuous

in general | not true any more

only continuous linear mappings $\underbrace{\text{are}}_{\text{usually}}$ interesting!

1.22. Theorem: Let $\mathcal{H}$ be a Hilbert space and $L : \mathcal{H} \to \mathbb{F}$ a linear functional. The following statements are equivalent.

(a) L is continuous.

(b) L is continuous at 0.

(c) There is a constant $c > 0$ s.th

$$|L(x)| \leq c \|x\| \qquad \forall x \in \mathcal{H}$$

<u>Proof</u>: (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c): Suppose L is continuous at 0, i.e.

$\exists \delta > 0$ s.th.

$$\|y\| \leq \delta \Rightarrow |L(y)| \leq 1 \qquad (\text{note}: L(0) = 0)$$

Consider now $x \in \mathcal{H}$, $x \neq 0$

Put $y := \frac{\delta}{\|x\|} \cdot x \quad \Rightarrow \|y\| = \delta$

$$\Rightarrow \underbrace{|L(y)|}_{= \frac{\delta}{\|x\|} |L(x)|} \leq 1$$

$$\Rightarrow |L(x)| \leq \frac{1}{\delta} \cdot \|x\|$$

Put $c := \frac{1}{\delta}$

(c) $\Rightarrow$ (a): Suppose $x_n \to x$, i.e. $\|x - x_n\| \to 0$

$$\Rightarrow \underbrace{|L(x - x_n)|}_{= |L(x) - L(x_n)|} \leq c \|x - x_n\| \to 0$$

$$\Rightarrow L(x_n) \to L(x)$$

i.e. L continuous. □

1.23. Def: 1) A linear functional $L: \mathcal{H} \to \mathbb{F}$ which fulfills the conditions in 1.22 is called bounded linear functional on $\mathcal{H}$.

2) The dual space of $\mathcal{H}$ is the set of all bounded linear functionals on $\mathcal{H}$.

Notation: $\mathcal{H}^*$

<u>1.24. Remark</u>: The dual space carries a canonical vector space structure,

$(L_1 + L_2)(x) := L_1(x) + L_2(x)$

$(\lambda L)(x) := \lambda \cdot L(x)$,

and a canonical norm

$\|L\| := \sup\{|L(x)| \mid \|x\| \leq 1\}$

(i.e. $\|L\|$ is the smallest c s.th. $|L(x)| \leq c\|x\| \;\forall x \in \mathcal{H}$)

<u>1.25. Theorem (Riesz Representation Theorem)</u>:

If $L \in \mathcal{H}^*$, then there is a unique $x_0 \in \mathcal{H}$ s.th.

$L(x) = \langle x, x_0 \rangle \qquad \forall x \in \mathcal{H}$.

Moreover, $\|L\| = \|x_0\|$.

<u>1.26 Remarks</u>: 1) The converse is trivial: Fix $x_0 \in \mathcal{H}$ and define $L(x) := \langle x, x_0 \rangle$

$\Rightarrow L$ linear and

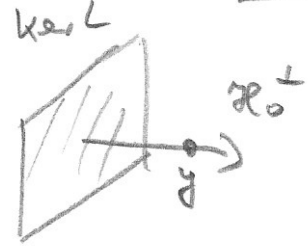
$|L(x)| = |\langle x, x_0 \rangle| \overset{C-S}{\leq} \|x\| \cdot \|x_0\|$

$\Rightarrow L \in \mathcal{H}^*$ and $\|L\| \leq \|x_0\|$

$L\left(\frac{x_0}{\|x_0\|}\right) = \frac{L(x_0)}{\|x_0\|} = \frac{\langle x_0, x_0 \rangle}{\|x_0\|} = \|x_0\| \Rightarrow \|L\| = \|x_0\|$

2) Since $\begin{array}{l} x_0 \mapsto L \\ \mathcal{H} \to \mathcal{H}^* \end{array}$ preserves also vector space structure,

we have: $\mathcal{H} \cong \mathcal{H}^*$



Proof: $L \equiv 0 \iff x_0 = 0$

consider now $L \not\equiv 0$

$\mathcal{H}_0 := \ker L$ closed linear subspace

$L \neq 0 \Rightarrow \mathcal{H}_0 \neq \mathcal{H} \Rightarrow \mathcal{H}_0^\perp \neq \{0\}$

$\Rightarrow \exists\, y \in \mathcal{H}_0^\perp$ s.th. $L(y) = 1$

Consider $x \in \mathcal{H} \Rightarrow L(x - L(x)y) = L(x) - L(x)\underbrace{L(y)}_{=1} = 0$

$\Rightarrow x - L(x)y \in \mathcal{H}_0$ (i.e. $\mathcal{H}_0^\perp = \mathbb{F}y$ one-dimensional)

$\Rightarrow \langle x - L(x)y, y\rangle = 0$

$\Rightarrow \langle x, y\rangle = L(x)\langle y, y\rangle$

$\Rightarrow L(x) = \frac{1}{\|y\|^2}\langle x, y\rangle$

put $x_0 := \frac{y}{\|y\|^2}$

Uniqueness: Assume $\langle x, x_0\rangle = \langle x, x_1\rangle \quad \forall x \in \mathcal{H}$

$\Rightarrow \langle x, x_0 - x_1\rangle = 0 \quad \forall x \in \mathcal{H}$

$\Rightarrow x_0 - x_1 \in \mathcal{H}^\perp = \{0\}$

$\Rightarrow x_0 = x_1$

$\|L\| = \|x_0\|$: compare 1.26.

1.27. Remark: Dirac's physical notation:

$$\left.\begin{array}{lll} x \in \mathcal{H} & \leftrightarrow & |x\rangle \\ L \in \mathcal{H}^* & \leftrightarrow & \langle x_0| \end{array}\right\} \Rightarrow L(x) = \langle x_0 | x \rangle = \langle x_0, x \rangle$$

note convention in physics: $\langle \cdot | \cdot \rangle$ linear in second variable

1.28. Def.: Let $I \neq \emptyset$ and $\mathcal{H}$ be a Hilbert space. A family $(e_i)_{i \in I} \subset \mathcal{H}$ is called an orthonormal set (ONS), if

$$\langle e_i, e_j \rangle = \delta_{ij} \qquad \forall i,j \in I$$

Aim: Represent every x in the form

$$x = \sum \langle x, e_i \rangle e_i$$

for ONS (e_i) which is "big enough"

1.29. Theorem (Bessel's Inequality): If $(e_n)_{n \in \mathbb{N}}$ is a countable ONS and $x \in \mathcal{H}$, then

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 .$$

Proof: Put $x_n := x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k$

$$\Rightarrow \langle x_n, e_i \rangle = \langle x, e_i \rangle - \underbrace{\sum_{k=1}^{n} \langle x, e_k \rangle \underbrace{\langle e_k, e_i \rangle}_{\delta_{ki}}}_{\langle x, e_i \rangle}$$

$$= 0$$

$$\Rightarrow x_n \perp e_i \qquad \forall\ 1 \leq i \leq n$$

$$\Rightarrow \|x\|^2 = \|x_n + \sum_{k=1}^{n} \langle x, e_k \rangle e_k\|^2$$

$$= \|x_n\|^2 + \sum_{k=1}^{n} \underbrace{\|\langle x, e_k \rangle e_k\|^2}_{|\langle x, e_k \rangle|^2}$$

$$\geq \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

$n \to \infty \quad \Rightarrow \quad$ assertion $\square$

1.30. Corollary: If $(e_i)_{i \in I}$ is an ONS in $\mathcal{H}$ and $x \in \mathcal{H}$, then $\langle x, e_i \rangle \neq 0$ for at most countably many $i \in I$.

Proof: $$\{i \in I : \langle x, e_i \rangle \neq 0\} = \bigcup_{n=1}^{\infty} \underbrace{\{i \in I : \langle x, e_i \rangle \geq \tfrac{1}{n}\}}_{\text{finite by } 1.29}$$

$\square$

1.31. Corollary: If $(e_i)_{i\in I}$ is an ONS in $\mathcal{H}$ and $x \in \mathcal{H}$, then

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \le \|x\|^2.$$

1.32. Def.: A Hilbert space is called <u>separable</u> if it contains a countable dense subset.

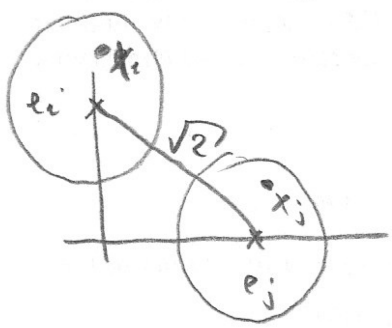
1.33. Lemma: If $\mathcal{H}$ is separable and $(e_i)_{i\in I}$ ONS in $\mathcal{H}$, then I is countable.

Proof: Let X be dense subset of $\mathcal{H}$

$\Rightarrow \forall i \in I \;\; \exists x_i \in X \; : \; \|x_i - e_i\| \le \frac{1}{2}$

note: $\|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j\rangle = 2$

$\Rightarrow \|e_i - e_j\| = \sqrt{2}$



$\Rightarrow x_i \ne x_j \quad \forall i \ne j$

thus: I uncountable

$\Rightarrow X$ uncountable

$\square$

<u>1.34. Theorem</u>: Let $(e_i)_{i\in I}$ be an ONS in a separable Hilbert space $\mathcal{H}$ (thus: I countable).
Then the following statements are equivalent:

(a) $\operatorname{span}\{e_i \mid i\in I\} := \{\sum_{\text{finite}} \alpha_i e_i \mid \alpha_i \in \mathbb{F}\}$

is dense in $\mathcal{H}$

(b) $x = \sum_{i\in I} \langle x, e_i\rangle e_i \qquad \forall x\in\mathcal{H}$

(c) $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 \qquad \forall x\in\mathcal{H}$

(Parseval's Identity)

<u>Proof</u>: (c) $\Rightarrow$ (b): Put $x_n := x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k$

to show: $x_n \to 0$

$$\|x\|^2 = \|x_n + \sum_{k=1}^{n}\langle x, e_k\rangle e_k\|^2$$

$$= \|x_n\|^2 + \underbrace{\sum_{k=1}^{n} |\langle x, e_k\rangle|^2}_{\to \|x\|^2} \qquad (\text{cf. } 1.29)$$

$\Rightarrow \|x_n\| \to 0$

(b) $\Rightarrow$ (a): clear, since every $x\in\mathcal{H}$ can be approximated by finite linear combinations in (e_i)

(a) $\Rightarrow$ (c): Fix $x \in \mathcal{H}$, $\varepsilon > 0$

$\overset{a)}{\Longrightarrow} \exists M$ s.th. $\operatorname{dist}(x, \mathcal{H}_M) < \varepsilon$

where $\mathcal{H}_M = \operatorname{span}\{e_1, \dots, e_M\}$

$$x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \perp \mathcal{H}_M$$

$$\overset{1.14}{\Longrightarrow} \| x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \| = \operatorname{dist}(x, \mathcal{H}_M) < \varepsilon$$

$$\Rightarrow \underset{\substack{\uparrow \\ 1.29 \\ \text{(Bessel's Ineq.)}}}{0 \leq} \|x\|^2 - \sum_{k=1}^{M} |\langle x, e_k \rangle|^2 = \| x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \|^2 < \varepsilon^2$$

$\varepsilon \searrow 0 \Rightarrow$ (c) $\square$

<u>1.35. Def</u>: An ONS in a separable Hilbert space which fulfills the conditions of 1.34 is called <u>complete ONS</u> or <u>orthonormal basis</u> (ONB).

<u>1.36. Remarks</u>: 1) note difference between basis from linear algebra (Hamel-basis) and ONB in Hilbert space theory.

Hamel basis $\{f_i\}$: $\forall x \; \exists \alpha_i$: only finitely many $\alpha_i \neq 0$

$$x = \sum_{\text{finite}} \alpha_i f_i$$

ONB $\{e_i\}$: $\forall x \; \exists \beta_i$:

$$x = \underset{\substack{\uparrow \\ \text{possibly infinite sum}}}{\sum} \beta_i e_i$$

2) Let $(e_i)_{i\in I}$ be ONB for $\mathcal{H}$

1.34 says: every $x \in \mathcal{H}$ can be written as

$$x = \sum_{i\in I} \alpha_i e_i \qquad \text{where } \alpha_i = \langle x, e_i \rangle$$

$$\text{and } \sum_{i\in I} |\alpha_i|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 = \|x\|^2 < \infty$$

- this representation is unique:

$$x = \sum_{i\in I} \alpha_i e_i \implies \alpha_i = \langle x, e_i \rangle$$

3) In particular:

$(e_i)_{i\in I}$ ONB $\implies$ e_i independent

because $0 = \sum \alpha_i e_i \implies \alpha_i = 0 \ \forall i$

2) $(e_i)_{i\in I}$ ONB $\Rightarrow$ e_i linearly independent:

Assume $0 = \sum \alpha_i e_i \Rightarrow \underset{\underset{0}{\|}}{\langle 0, e_j\rangle} = \sum \alpha_i \underbrace{\langle e_i, e_j\rangle}_{\delta_{ij}} = \alpha_j$

$\Rightarrow \alpha_j = 0 \quad \forall j$

1.37. Theorem (Gram-Schmidt Orthogonalization Process):

Let $(x_n)_{n=1}^N$ (where $N \in \mathbb{N} \cup \{\infty\}$) be a sequence of linearly independent elements in a Hilbert space $\mathcal{H}$. Then there is an ONS $(e_n)_{n=1}^N$ in $\mathcal{H}$ such that

$$\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{e_1, \dots, e_n\} \qquad \forall n = 1, \dots, N$$

Proof: $e_1 := \dfrac{x_1}{\|x_1\|}$

assume $e_1, \dots, e_n$ found $\rightsquigarrow x'_{n+1} := x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k\rangle e_k$

$\Rightarrow x'_{n+1} \perp e_k \quad \forall\, 1 \le k \le n$

$$\underset{\substack{\uparrow\\ \text{linearly indep.}}}{x_{n+1} \notin} \operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{e_1, \dots, e_n\} \Rightarrow x'_{n+1} \neq 0$$

Put $e_{n+1} := \dfrac{x'_{n+1}}{\|x'_{n+1}\|}$

$\square$

<u>1.38. Corollary:</u> Every separable Hilbert space has a countable ONB.

<u>Proof:</u> $\mathcal{H}$ separable $\Rightarrow \exists$ dense sequence $(\xi_j)_{j \in \mathbb{N}}$

Cancel linearly dependent elements

$\Rightarrow \exists$ linearly independent sequence $(x_n)_{n=1}^{N}$

$(N \in \mathbb{N}$ or $N = \infty)$ s.th. $\operatorname{span}\{x_1, x_2, \dots\}$ dense in $\mathcal{H}$

$\overset{G.S.}{\Longrightarrow} \exists$ ONS $(e_n)_{n=1}^{N}$ with

$\operatorname{span}\{e_1, \dots, e_n\} = \operatorname{span}\{x_1, \dots, x_n\} \quad \forall n$

$\Rightarrow \operatorname{span}\{e_1, e_2, \dots\} = \operatorname{span}\{x_1, x_2, \dots\}$ dense in $\mathcal{H}$

$\overset{1.34 + 1.35}{\Longrightarrow} (e_n)_{n=1}^{N}$ ONB $\square$

<u>1.39. Prop.:</u> Let $\mathcal{H}$ be a separable Hilbert space and $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ two ONB. Then I and J have the same cardinality, i.e.

$|I| = |J| \in \mathbb{N}$ or $|I| = |J| = \infty$ $(= \aleph_0 =$ countable infinity$)$

<u>Proof:</u> Assume $|J| < \infty$

$\Rightarrow e_i = \sum_{j=1}^{|J|} \alpha_{ij} f_j$

$\overset{\text{linear algebra}}{\Longrightarrow} |I| \leq |J|$

$\overset{\text{symm.}}{\Longrightarrow} |I| = |J|$

Assume $|I|, |J|$ infinite

$\overset{1.38}{\Longrightarrow}$ $|I| =$ countably infinite $= |J|$ $\square$

<u>1.40. Definition:</u> The cardinality of an ONB of $\mathcal{H}$ is called <u>dimension of $\mathcal{H}$</u>.

Notation: $\dim \mathcal{H}$

<u>1.41. Examples:</u> 1) $\mathcal{H} = \mathbb{F}^n = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \mathbb{F}\}$

$(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ ONB

$\dim(\mathbb{F}^n) = n$

2) $\mathcal{H} = \ell_2 = \{(\alpha_i)_{i=1}^\infty \mid \sum_{i=1}^\infty |\alpha_i|^2 < \infty\}$

$e_i := (0, \dots, 0, 1, 0, \dots)$

$\uparrow$ i-th position

$\Rightarrow (e_i)_{i=1}^\infty$ ONB $\quad \left((\alpha_i)_{i=1}^\infty = \sum_{i=1}^\infty \alpha_i e_i \right)$

$\Rightarrow \dim(\ell_2) = \infty$

<u>1.42. Def.:</u> Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbertspaces.

An <u>isomorphism</u> between $\mathcal{H}$ and $\mathcal{K}$ is a linear mapping $\mathcal{U}: \mathcal{H} \to \mathcal{K}$ s.th.

i) $\mathcal{U}$ surjectiv $\qquad$ $\mathcal{U}$ isometric, i.e

ii) $\langle \mathcal{U}x, \mathcal{U}y \rangle_{\mathcal{K}} = \langle x, y \rangle_{\mathcal{H}} \quad \forall x, y \in \mathcal{H}$.

$\mathcal{H}$ and $\mathcal{K}$ are <u>isomorphic</u> if such an isomorphism exists

1.43. Remarks: 1) Assume $\mathcal{U}x = \mathcal{U}y$

$\Rightarrow \mathcal{U}(x-y) = 0$

$\Rightarrow 0 = \langle \mathcal{U}(x-y), \mathcal{U}(x-y) \rangle$

$= \langle x-y, x-y \rangle$

$= \|x-y\|^2$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\Rightarrow \mathcal{U}$ injective

thus: isomorphism is bijective

$\Rightarrow \mathcal{U}^{-1} : \mathcal{H} \to \mathcal{H}$ exists

It is an isomorphism, too:

$x = \mathcal{U}^{-1}v, \quad y = \mathcal{U}^{-1}w$

$$\Rightarrow \langle \mathcal{U}x, \mathcal{U}y \rangle = \langle x, y \rangle = \langle \mathcal{U}^{-1}v, \mathcal{U}^{-1}w \rangle$$

$$\langle \mathcal{U}x, \mathcal{U}y \rangle = \langle \mathcal{U}\mathcal{U}^{-1}v, \mathcal{U}\mathcal{U}^{-1}w \rangle = \langle v, w \rangle$$

2) Note: if $\underbrace{\dim < \infty,}_{\text{and } \mathcal{H} = \mathcal{H}}$ then every isometry is automatically surjective.

Not true for $\dim = \infty$

1.44. Example: $S: l_2 \to l_2$ one-sided shift

$$(\alpha_1, \alpha_2, \dots) \mapsto (0, \alpha_1, \alpha_2, \dots)$$

S is isometry: $\langle Sx, Sy \rangle = \langle x, y \rangle \quad \forall x, y \in l_2$

but not surjective: there is no $x \in l_2$ s.th.

$$Sx = (1, 0, 0, \dots)$$

(compare "Hilbert's hotel")

1.45. Theorem: Two separable Hilbert spaces are isomorphic if and only if they have the same dimension.

Proof: "$\Rightarrow$": Let $\mathcal{U}: \mathcal{H}_1 \to \mathcal{H}_2$ be isomorphism

If $(e_i)_{i \in I}$ is ONB of $\mathcal{H}_1$

$\Rightarrow (\mathcal{U} e_i)_{i \in I}$ ONB of $\mathcal{H}_2$

$\Rightarrow \dim \mathcal{H}_2 = |I| = \dim \mathcal{H}_1$

"$\Leftarrow$": Assume $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = |I|$

$\Rightarrow \exists$ ONB $(e_i)_{i \in I}$ of $\mathcal{H}_1$

$\exists$ ONB $(f_i)_{i \in I}$ of $\mathcal{H}_2$

Def. $\mathcal{U}: \mathcal{H}_1 \to \mathcal{H}_2$ by

$$\mathcal{U}\left(\sum_{i \in I} \alpha_i e_i\right) = \sum_{i \in I} \alpha_i f_i$$

every $x \in \mathcal{H}_1$ uniquely representable in this form, comp. 1.36.

$\mathcal{U}$ is linear, surjective and isometry:

$$\langle U(\sum_i \alpha_i e_i), U(\sum \beta_j e_j)\rangle = \sum_{i,j} \langle \alpha_i f_i, \beta_j f_j \rangle$$

$$= \sum_{i,j} \alpha_i \overline{\beta_j} \underbrace{\langle f_i, f_j \rangle}_{\delta_{ij}}$$

$$= \sum \alpha_i \overline{\beta_i}$$

$$= \langle \sum_i \alpha_i e_i, \sum_j \beta_j e_j \rangle \qquad \square$$

<u>1.46. Corollary</u>: A separable Hilbert space is isomorphic to $\mathbb{F}^n$ for some $n \in \mathbb{N}$ (finite dimensional case) or to $\mathbb{F}^\infty := \ell_2$ (infinite dimensional case).