1. Hilbert spaces

$\mathbb{F}$ field, usually $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

1.1. Def.: A <u>vector space over $\mathbb{F}$</u> is a set V with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ such that

(i) $(V, +)$ is abelian group

(ii) $\lambda(x+y) = \lambda x + \lambda y \quad \forall \lambda \in \mathbb{F},\ x, y \in V$

(iii) $(\lambda + \mu) x = \lambda x + \mu x \quad \forall \lambda, \mu \in \mathbb{F},\ x \in V$

(iv) $(\lambda \mu) x = \lambda(\mu x) \quad \forall \lambda, \mu \in \mathbb{F},\ x \in V$

(v) $1 . x = x \quad \forall x \in V$

1.2. Def.: 1) If V is a vector space over $\mathbb{F}$, an <u>inner product</u> on V is a function

$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$

with the following properties:

(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle \quad \forall \lambda, \mu \in \mathbb{F}$
$\forall x, y, z \in V$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in V$

(iii) $\langle x, x \rangle \geq 0 \quad \forall x \in V$

(iv) $\langle x, x \rangle = 0 \iff x = 0$

2) A <u>pre-Hilbert space</u> is a vector space together with an inner product.

1.3. Examples: 1) $V = \mathbb{F}^n$

$x = (\alpha_1, \dots, \alpha_n) \qquad \alpha_1, \dots, \alpha_n \in \mathbb{F}$

$\hat{=} \alpha_1 \vec{e_1} + \dots + \alpha_n \vec{e_n}$

$$\langle (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \rangle = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

remind: $\left| \sum_{i=1}^{n} \alpha_i \overline{\beta_i} \right| \le \left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{n} |\beta_i|^2 \right)^{1/2}$

Cauchy-Schwarz inequality

2) $V = L^2(0,1) = \{ f \mid \int_0^1 |f(t)|^2 dt < \infty \}$

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, dt$$

note: $f, g \in L^2 \Rightarrow$

$$\left| \int_0^1 f(t) \bar{g}(t) \, dt \right| \le \left(\int_0^1 |f(t)|^2 dt \right)^{1/2} \cdot \left(\int_0^1 |g(t)|^2 dt \right)^{1/2}$$

Hölder's inequality

1.4. Theorem (Cauchy-Schwarz Inequality)

1) If $\langle \cdot , \cdot \rangle$ is an inner product on V, then

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle \cdot \langle y,y\rangle \qquad \forall x,y \in V$$

2) We have "=" if and only if

x and y are linearly dependent

Proof: 1) We have $\forall \lambda \in \mathbb{F}$:

$$0 \leq \langle \lambda x + y , \lambda x + y\rangle$$

$$= \lambda\bar{\lambda} \langle x,x\rangle + \lambda\langle x,y\rangle + \bar{\lambda}\langle y,x\rangle + \langle y,y\rangle$$

If $\langle x,x\rangle = 0 \implies \langle x,y\rangle = 0 \quad \forall y \in V$

(i.e. $x = 0$) $\implies$ Ineq. true

otherwise put $\lambda = -\dfrac{\langle y,x\rangle}{\langle x,x\rangle}$

$$\implies 0 \leq \underbrace{\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle} - 2\,\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}}_{-\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}} + \langle y,y\rangle$$

$$\implies \langle y,y\rangle \geq \frac{|\langle x,y\rangle|^2}{\langle x,x\rangle}$$

2) $\text{"="} \iff \left\{ \begin{array}{l} x = 0 \text{ or} \\ \langle \lambda x+y , \lambda x+y\rangle = 0 \iff \lambda x + y = 0 \end{array} \right\} \iff$ lin. dep. $\square$