We will put

$$\|x\| := \sqrt{\langle x, x\rangle} \qquad \text{"length" of } x$$

$V = \mathbb{F}^n$: $\quad x = (\alpha_1, \dots, \alpha_n)$

$$\Rightarrow \|x\| = \sqrt{|\alpha_1|^2 + \dots + |\alpha_n|^2}$$

<u>1.5. Def:</u> 1) If V is a vector space over $\mathbb{F}$, a <u>norm</u> on V is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that

(i) $\|\lambda x\| = |\lambda| \, \|x\| \qquad \forall \lambda \in \mathbb{F}, x \in V$

(ii) $\|x+y\| \leq \|x\| + \|y\| \qquad \forall x, y \in V$

("triangle inequality")

(iii) $\|x\| = 0 \iff x = 0$

2) A <u>normed space</u> is a vector space together with a norm.

<u>1.6. Proposition:</u> If V is a pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then

$$\|x\| := \sqrt{\langle x, x\rangle} \qquad (x \in V)$$

defines a norm on V.

Thus: a pre-Hilbert space is a normed space

<u>Proof:</u> - $\|x\| \geq 0$ ✓

- $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda \bar{\lambda} \langle x, x \rangle} = |\lambda| \, \|x\|$

- $\|x\| = 0 \iff \langle x, x \rangle = 0 \iff x = 0$

- $$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2 \underbrace{\mathrm{Re} \langle x, y \rangle}_{\substack{\leq |\langle x, y \rangle| \\ \leq \|x\| \cdot \|y\| \quad (\text{C-S})}} + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$ □

<u>1.7. Def.:</u> 1) A normed space V is <u>complete</u>, if every Cauchy sequence converges, i.e.

$$\left. \begin{array}{l} (x_n) \text{ CS}: \; x_n \in V \text{ and} \\ \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that} \\ \|x_n - x_m\| \leq \varepsilon \quad \forall n, m \geq N \end{array} \right\} \Rightarrow \begin{array}{l} \exists x \in V \text{ such that } x_n \to x \\ (\text{i.e.: } \forall \varepsilon > 0 \; \exists N \in \mathbb{N}: \\ \quad \|x_n - x\| \leq \varepsilon \quad \forall n \geq N) \\ (\text{i.e.: } \|x_n - x\| \to 0) \end{array}$$

2) A <u>Hilbert space</u> $\mathcal{H}$ is a pre-Hilbert space that is complete (with respect to the norm $\|x\| = \sqrt{\langle x, x \rangle}$).

3) A <u>Banach space</u> X is a normed space that is complete.

thus: a Hilbert space is a Banach space

1.8. Examples: 1) $\mathbb{F}^n$, $L^2(0,1)$ are Hilbert spaces

2) $l_2 \cong \mathbb{F}^\infty$

$$l_2 := \{ (\alpha_i)_{i=1}^\infty \mid \alpha_i \in \mathbb{F}, \sum_{i=1}^\infty |\alpha_i|^2 < \infty \}$$

space of square-summable sequences

$$\langle (\alpha_i), (\beta_j) \rangle := \sum_{i=1}^\infty \alpha_i \overline{\beta_i}$$

l_2 is Hilbert space

3) Each closed linear subspace of Hilbertspace (Banach space) is a Hilbert space (Banach space).

4) $\mathcal{C}[a,b] := \{ f : [a,b] \to \mathbb{F} \mid f \text{ continuous} \}$

$$\|f\| := \sup_{t \in [a,b]} |f(t)|$$

is Banach space, but not Hilbert space.

5) $\mathcal{C}[a,b]$ is w.r.t. $\langle f, g \rangle := \int_a^b f(t)\overline{g(t)}\,dt$

pre-Hilbert space, but not complete.

$$\left(\overline{\mathcal{C}[a,b]}^{\langle \cdot,\cdot \rangle} = L^2[a,b] \right)$$

<u>1.9. Prop.:</u> 1) Let X be a Banach space. Then the maps (for $\alpha \in \mathbb{F}$)

$x \mapsto \|x\|$, $(x, y) \mapsto x+y$, $x \mapsto \alpha x$ are continuous

2) Let $\mathcal{H}$ be a Hilbert space. Then, for fixed $y \in \mathcal{H}$, the maps $x \mapsto \langle x, y \rangle$ and $x \mapsto \langle y, x \rangle$ are continuous.

<u>Proof:</u> 1) $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$$\left.\begin{array}{l} \Rightarrow \|x\| - \|y\| \leq \|x-y\| \\ \overset{x \leftrightarrow y}{\Rightarrow} \|y\| - \|x\| \leq \|x-y\| \end{array}\right\} \Rightarrow \big| \|x\| - \|y\| \big| \leq \|x-y\|$$

thus: $x_n \to x$ (i.e. $\|x_n - x\| \to 0$)

$\Rightarrow \big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| \to 0$

$\Rightarrow \|x_n\| \to \|x\|$

(*)

2) $x_n \to x \Rightarrow |\langle x_n, y\rangle - \langle x, y\rangle| =$

$$= |\langle x_n - x, y\rangle| \overset{CS}{\leq} \underbrace{\|x_n - x\|}_{\to 0} \cdot \|y\|$$

$\Rightarrow \langle x_n, y\rangle \to \langle x, y\rangle$

□

(*) $x_n \to x$, $y_n \to y$

$\Rightarrow \|(x_n + y_n) - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$

$\Rightarrow x_n + y_n \to x + y$

1.10. Def.: Let V be a pre-Hilbert space.

1) $x, y \in V$ are orthogonal ($x \perp y$), if

$$\langle x, y \rangle = 0$$

2) Subsets $A, B \subset V$ are orthogonal ($A \perp B$),

if $\langle x, y \rangle = 0 \quad \forall x \in A, y \in B$

3) If $A \subset V$, then

$$A^\perp := \{ x \in V \mid x \perp A \}$$

is called orthogonal complement of A.

1.11. Theorem: Let V be a pre-Hilbert space.

Then we have:

1) Polarisation Identity

$$\|x+y\|^2 = \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2$$

2) Parallelogram Law:

$$\|x+y\|^2 + \|x-y\|^2 = 2 (\|x\|^2 + \|y\|^2)$$

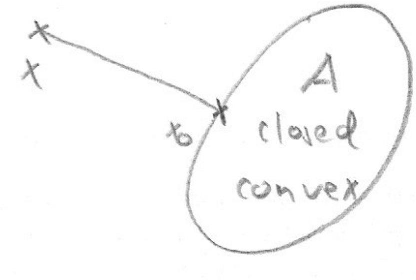
3) Pythagorean Theorem:

If $x_1, \dots, x_n \in V$ such that $x_i \perp x_j$ ($i \neq j$),

then

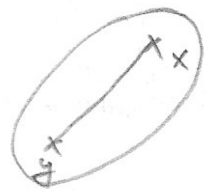
$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2$$

Proof: Exercise!

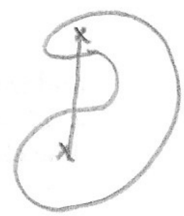


1.12. Def.: If V is a vector space over $\mathbb{F}$ and $A \subset V$, then A is <u>convex</u>, if

$$\left.\begin{array}{l} x, y \in A \\ 0 \le t \le 1 \end{array}\right\} \Rightarrow tx + (1-t)y \in A$$



convex


not convex

1.13. Theorem: Let $A \neq \emptyset$ be a closed convex subset of a Hilbert space $\mathcal{H}$. Then, for each $x \in \mathcal{H}$, there is a unique point $x_0 \in A$ such that

$$\|x - x_0\| = \operatorname{dist}(x, A) := \inf_{y \in A} \|x - y\|$$

Proof: By def. of $\operatorname{dist}(x, A)$: $\exists\, y_n \in \mathcal{H}$ such that

$\|x - y_n\| \to \operatorname{dist}(x, A)$

want to show: y_n converges

suffices: $\{y_n\}$ Cauchy sequence