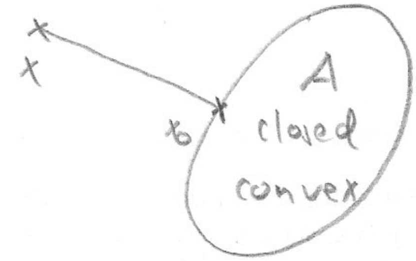
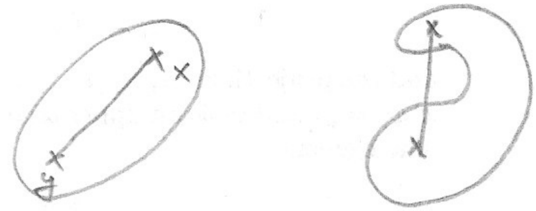


<u>1.12. Def.</u>: If V is a vector space over $\mathbb{F}$ and $A \subset V$, then A is <u>convex</u>, if

$$\left.\begin{array}{l} x, y \in A \\ 0 \le t \le 1 \end{array}\right\} \Rightarrow tx + (1-t)y \in A$$



convex　　　not convex

<u>1.13. Theorem</u>: Let $A \neq \emptyset$ be a closed convex subset of a Hilbert space $\mathcal{H}$. Then, for each $x \in \mathcal{H}$, there is a unique point $x_0 \in A$ such that

$$\|x - x_0\| = \operatorname{dist}(x, A) := \inf_{y \in A} \|x - y\|$$

<u>Proof</u>: By def. of $\operatorname{dist}(x, A)$: $\exists\, y_n \in \overset{A}{\cancel{\mathcal{H}}}$ such that

$$\|x - y_n\| \to \operatorname{dist}(x, A)$$

want to show: y_n converges

suffices: $\{y_n\}$ Cauchy sequence

use Parallelogram Law for $x-y_n$ and $x-y_m$

$$\Rightarrow \|x-y_n+x-y_m\|^2+\|x-y_n-(x-y_m)\|^2 =$$

$$= 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right)$$

$$\Rightarrow \|y_m-y_n\|^2 = 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right) - \underbrace{4\|x-\underbrace{\tfrac{y_n+y_m}{2}}_{\in A}\|^2}_{\geq \operatorname{dist}(x,A)^2}$$

$$\leq 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right) - 4\operatorname{dist}(x,A)^2$$

$\leq \varepsilon$ for n,m sufficiently large

$\Rightarrow \{y_n\}$ CS

$\Rightarrow \exists x_0 \in \mathcal{H}$ s.th. $y_n \to x_0$ (i.e. $\|x_0-y_n\| \to 0$)

A closed $\Rightarrow x_0 \in A$

$$\operatorname{dist}(x,A) \underset{\substack{\nearrow \\ x_0\in A}}{\leq} \|x-x_0\| = \|x-y_n+y_n-x_0\|$$

$$\leq \underset{\substack{\downarrow \\ \operatorname{dist}(x,A)}}{\|x-y_n\|} + \underset{\substack{\downarrow \\ 0}}{\|y_n-x_0\|}$$

$\Rightarrow \|x-x_0\| = \operatorname{dist}(x,A)$ $\square$

Uniqueness: Assume $\|x-x_1\| = \operatorname{dist}(x,A) = \|x-x_0\|$

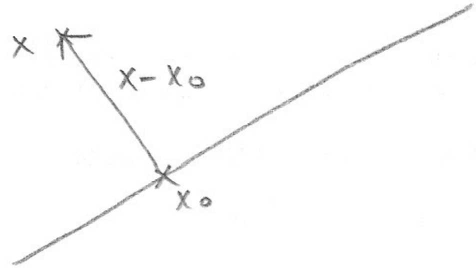
$\Rightarrow (y_n) = (x_0, x_1, x_0, x_1, \dots)$ is sequence as above $\Rightarrow (y_n)$ converges $\Rightarrow x_0 = x_1$

Consider now

$A = \mathcal{H}_0$ sub Hilbert space

(closed linear subspace)

$A = \mathcal{H}_0$ (note: linear $\Rightarrow$ convex)



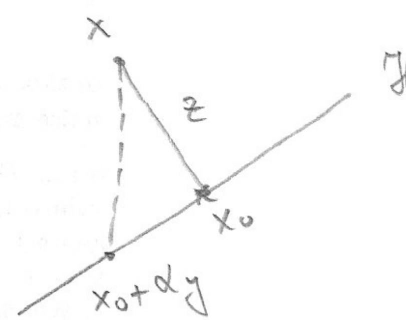
<u>1.14. Theorem</u>: Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed linear subspace.

Then, for $x \in \mathcal{H}$ and $x_0 \in \mathcal{H}_0$, the following are equivalent:

(a) $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$

(b) $x - x_0 \perp \mathcal{H}_0$

<u>Proof</u>: (a) $\Rightarrow$ (b): Consider $y \in \mathcal{H}_0$



to show: $\langle x - x_0, y \rangle = 0$

suffices to restrict to $\|y\| = 1$

Put $z := x - x_0$, then $\forall \alpha \in \mathbb{F}$:

$$\|z\|^2 = \operatorname{dist}(x, \mathcal{H}_0)^2 \leq \|x - \underbrace{(x_0 + \alpha y)}_{\in \mathcal{H}_0}\|^2$$

$$= \|z - \alpha y\|^2$$

$$= \langle z - \alpha y, z - \alpha y \rangle$$

$$= \langle z, z \rangle - \alpha \langle y, z \rangle - \bar{\alpha} \langle z, y \rangle + |\alpha|^2 \underbrace{\langle y, y \rangle}_{1}$$

Choose $\alpha = \langle z, y \rangle$: $0 \leq -|\alpha|^2 \Rightarrow \underset{\langle z, y \rangle}{\overset{\alpha}{\shortparallel}} = 0$

b) $\Rightarrow$ a) $x - x_0 \perp \mathcal{H}_0$

to show: $\|x - x_0\| \leq \|x - y\| \quad \forall y \in \mathcal{H}_0$

Consider $y \in \mathcal{H}_0$:

$$\|x-y\|^2 = \|\underbrace{(x - x_0)}_{\perp \mathcal{H}_0} + \underbrace{(x_0 - y)}_{\in \mathcal{H}_0}\|^2$$

$$\overset{\text{Pythagoras}}{=} \|x - x_0\|^2 + \|x_0 - y\|^2$$

$$\geq \|x - x_0\|^2$$

□

1.15. Def.: Let $\mathcal{H}_0, \mathcal{H}_1 \subseteq \mathcal{H}$ be closed linear subspaces of a Hilbertspace $\mathcal{H}$ such that $\mathcal{H}_0 \perp \mathcal{H}_1$. Then

$$\mathcal{H}_0 \oplus \mathcal{H}_1 := \mathcal{H}_0 + \mathcal{H}_1 = \{x + y \mid x \in \mathcal{H}_0, y \in \mathcal{H}_1\} \subseteq \mathcal{H}$$

is called (orthogonal) direct sum of $\mathcal{H}_0$ and $\mathcal{H}_1$.

1.16. Remark: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ means:

- each $x \in \mathcal{H}$ can be written as

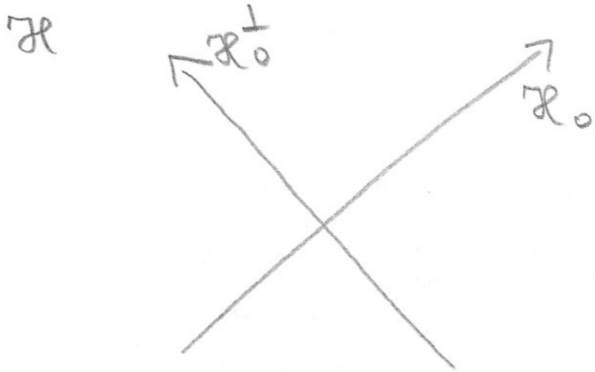
  $x = x_0 + x_1$ where $x_0 \in \mathcal{H}_0$, $x_1 \in \mathcal{H}_1$

- this representation is unique

  $x = x_0 + x_1 = x_0' + x_1'$

  $$\Rightarrow \underbrace{x_0 - x_0'}_{\in \mathcal{H}_0} = \underbrace{x_1' - x_1}_{\in \mathcal{H}_1} \in \mathcal{H}_0 \cap \mathcal{H}_1 \overset{(*)}{=} \{0\}$$

(*): $\mathcal{H}_0 \perp \mathcal{H}_1$: $x \in \mathcal{H}_0 \cap \mathcal{H}_1$: $\langle \underset{\mathcal{H}_0}{x}, \underset{\mathcal{H}_1}{x} \rangle = 0 \Rightarrow x = 0$



<u>1.17. Theorem:</u> Let $\mathcal{H}_0 \subset \mathcal{H}$ be a closed linear subspace. Then we have:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

<u>Proof:</u> (*) $\mathcal{H}_0 \perp \mathcal{H}_0^\perp$ clear

remains to show: $\mathcal{H}_0 + \mathcal{H}_0^\perp = \mathcal{H}$

Consider $x \in \mathcal{H}$

$\overset{1.12}{\Longrightarrow}$ $\exists\, x_0 \in \mathcal{H}_0$ s.th. $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$

$\overset{1.14}{\Longrightarrow}$ $x - x_0 \perp \mathcal{H}_0$, i.e. $x - x_0 \in \mathcal{H}_0^\perp$

$$\Rightarrow \quad x = \underset{\mathcal{H}_0}{\underset{\cap}{x_0}} + \underset{\mathcal{H}_0^\perp}{\underset{\cap}{(x - x_0)}}$$

□

(*) note: $A^\perp$ is always closed, for any $A \subset \mathcal{H}$;
$\mathcal{H}_0^\perp$ is also linear subspace

<u>1.18. Theorem:</u> Let $\mathcal{H}_0 \subset \mathcal{H}$ be closed linear subspace.
For $x \in \mathcal{H}$, define $Px = x_0$ where $x_0 \in \mathcal{H}_0$ with
$\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$ $(\Leftrightarrow x - x_0 \perp \mathcal{H}_0)$

Then

1) $P: \mathcal{H} \to \mathcal{H}$ is linear mapping

2) $\|Px\| \leq \|x\|$ $\forall x \in \mathcal{H}$

3) $P^2 = P$ $(P^2 x = P(Px))$

4) $\ker P = \mathcal{H}_0^\perp$, $\operatorname{ran} P = \mathcal{H}_0$

($\ker P = \{x \in \mathcal{H} \mid Px = 0\}$ kernel
$\operatorname{ran} P = \{y \in \mathcal{H} \mid y = Px \text{ for some } x \in \mathcal{H}\}$ range)

<u>Proof:</u> Exercise!

<u>1.19. Notation:</u> P is called <u>orthogonal projection</u>

from $\mathcal{H}$ onto $\mathcal{H}_0$.

1.20. Def.: A <u>linear functional</u> is a map

$L : \mathcal{H} \to \mathbb{F}$ such that

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \qquad \forall \alpha, \beta \in \mathbb{F}, \; x, y \in \mathcal{H}$$

1.21. Remark: in finite dimensions: linear $\Rightarrow$ continuous

in general | not true any more

only continuous linear mappings $\underbrace{\text{are}}_{\text{usually}}$ interesting!

1.22. Theorem: Let $\mathcal{H}$ be a Hilbert space and $L : \mathcal{H} \to \mathbb{F}$ a linear functional. The following statements are equivalent.

(a) L is continuous.

(b) L is continuous at 0.

(c) There is a constant $c > 0$ s.th

$$|L(x)| \leq c \, \|x\| \qquad \forall x \in \mathcal{H}$$

<u>Proof:</u> (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c): Suppose L is continuous at 0, i.e.

$\exists \, \delta > 0$ s.th.

$$\|y\| \leq \delta \Rightarrow |L(y)| \leq 1 \qquad (\text{note}: L(0) = 0)$$

Consider now $x \in \mathcal{H}$, $x \neq 0$

Put $y := \frac{\delta}{\|x\|} \cdot x \quad \Rightarrow \|y\| = \delta$

$$\Rightarrow |L(y)| \leq 1$$
$$\| $$
$$\frac{\delta}{\|x\|} |L(x)|$$

$$\Rightarrow |L(x)| \leq \frac{1}{\delta} \cdot \|x\|$$

Put $c := \frac{1}{\delta}$

(c) $\Rightarrow$ (a): Suppose $x_n \to x$, i.e. $\|x - x_n\| \to 0$

$$\Rightarrow |L(x - x_n)| \leq c \|x - x_n\| \to 0$$
$$\|$$
$$|L(x) - L(x_n)|$$

$$\Rightarrow L(x_n) \to L(x)$$

i.e. L continuous. □

<u>1.23. Def</u>: 1) A linear functional $L: \mathcal{H} \to \mathbb{F}$ which fulfills the conditions in 1.22 is called <u>bounded linear functional</u> on $\mathcal{H}$.

2) The <u>dual space</u> of $\mathcal{H}$ is the set of all bounded linear functionals on $\mathcal{H}$.

Notation: $\mathcal{H}^*$