1.24. Remark: The dual space carries a canonical vector space structure,

$$(L_1 + L_2)(x) := L_1(x) + L_2(x)$$

$$(\lambda L)(x) := \lambda \cdot L(x),$$

and a canonical norm

$$\|L\| := \sup\{|L(x)| \mid \|x\| \leq 1\}$$

(i.e. $\|L\|$ is the smallest c s.th. $|L(x)| \leq c\|x\|$ $\forall x \in \mathcal{H}$)

1.25. Theorem (Riesz Representation Theorem):

If $L \in \mathcal{H}^*$, then there is a unique $x_0 \in \mathcal{H}$ s.th.

$$L(x) = \langle x, x_0 \rangle \qquad \forall x \in \mathcal{H}.$$

Moreover, $\|L\| = \|x_0\|$.

1.26. Remarks: 1) The converse is trivial: Fix $x_0 \in \mathcal{H}$ and define $L(x) := \langle x, x_0 \rangle$

$\Rightarrow$ L linear and

$$|L(x)| = |\langle x, x_0 \rangle| \overset{C-S}{\leq} \|x\| \cdot \|x_0\|$$

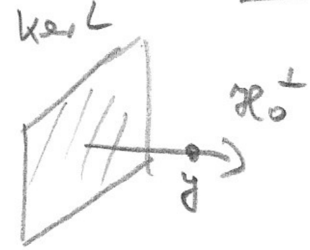
$\Rightarrow$ $L \in \mathcal{H}^*$ and $\|L\| \leq \|x_0\|$

$$L\left(\frac{x_0}{\|x_0\|}\right) = \frac{L(x_0)}{\|x_0\|} = \frac{\langle x_0, x_0 \rangle}{\|x_0\|} = \|x_0\| \quad \Rightarrow \|L\| = \|x_0\|$$

2) Since $\begin{array}{c} x_0 \mapsto L \\ \mathcal{H} \to \mathcal{H}^* \end{array}$ preserves also vector space structure,

we have: $\mathcal{H} \cong \mathcal{H}^*$

<u>Proof:</u> $L \equiv 0 \iff x_0 = 0$



consider now $L \not\equiv 0$

$\mathcal{H}_0 := \ker L$ closed linear subspace

$L \neq 0 \Rightarrow \mathcal{H}_0 \neq \mathcal{H} \Rightarrow \mathcal{H}_0^\perp \neq \{0\}$

$\Rightarrow \exists y \in \mathcal{H}_0^\perp$ s.th. $L(y) = 1$

Consider $x \in \mathcal{H} \Rightarrow L(x - L(x)y) = L(x) - L(x)\underbrace{L(y)}_{=1} = 0$

$\Rightarrow x - L(x)y \in \mathcal{H}_0$ (i.e. $\mathcal{H}_0^\perp = \mathbb{F}y$ one-dimensional)

$\Rightarrow \langle x - L(x)y, y \rangle = 0$

$\Rightarrow \langle x, y \rangle = L(x) \langle y, y \rangle$

$\Rightarrow L(x) = \frac{1}{\|y\|^2} \langle x, y \rangle$

put $x_0 := \frac{y}{\|y\|^2}$

Uniqueness: Assume $\langle x, x_0 \rangle = \langle x, x_1 \rangle \quad \forall x \in \mathcal{H}$

$\Rightarrow \langle x, x_0 - x_1 \rangle = 0 \quad \forall x \in \mathcal{H}$

$\Rightarrow x_0 - x_1 \in \mathcal{H}^\perp = \{0\}$

$\Rightarrow x_0 = x_1$

$\|L\| = \|x_0\|$: compare 1.26.

1.27. Remark: Dirac's physical notation:

$$\left.\begin{array}{lcl} x \in \mathcal{H} & \leftrightarrow & |x\rangle \\ L \in \mathcal{H}^* & \leftrightarrow & \langle x_0| \end{array}\right\} \Rightarrow L(x) = \langle x_0 | x \rangle = \langle x_0, x \rangle$$

note convention in physics: $\langle \cdot | \cdot \rangle$ linear in second variable

1.28 Def.: Let $I \neq \emptyset$ and $\mathcal{H}$ be a Hilbert space.
A family $(e_i)_{i \in I} \subset \mathcal{H}$ is called an orthonormal set (ONS), if

$$\langle e_i, e_j \rangle = \delta_{ij} \qquad \forall i,j \in I$$

Aim: Represent every x in the form

$$x = \sum \langle x, e_i \rangle e_i$$

for ONS (e_i) which is "big enough"

1.29. Theorem (Bessel's Inequality): If $(e_n)_{n \in \mathbb{N}}$ is a countable ONS and $x \in \mathcal{H}$, then

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 .$$

Proof: Put $x_n := x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k$

$$\Rightarrow \langle x_n, e_i \rangle = \underbrace{\langle x, e_i \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \underbrace{\langle e_k, e_i \rangle}_{\delta_{ki}}}_{\langle x, e_i \rangle}$$

$$= 0$$

$$\Rightarrow x_n \perp e_i \qquad \forall\ 1 \le i \le n$$

$$\Rightarrow \|x\|^2 = \|x_n + \sum_{k=1}^{n} \langle x, e_k \rangle e_k\|^2$$

$$= \|x_n\|^2 + \sum_{k=1}^{n} \underbrace{\|\langle x, e_k \rangle e_k\|^2}_{|\langle x, e_k \rangle|^2}$$

$$\ge \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

$n \to \infty \quad \Rightarrow \quad$ assertion $\square$

1.30. Corollary: If $(e_i)_{i \in I}$ is an ONS in $\mathcal{H}$ and $x \in \mathcal{H}$, then $\langle x, e_i \rangle \neq 0$ for at most countably many $i \in I$.

Proof: $$\{i \in I : \langle x, e_i \rangle \neq 0\} = \bigcup_{n=1}^{\infty} \underbrace{\{i \in I : \langle x, e_i \rangle \ge \tfrac{1}{n}\}}_{\text{finite by 1.29}}$$

$\square$

1.31. Corollary: If $(e_i)_{i\in I}$ is an ONS in $\mathcal{H}$ and $x \in \mathcal{H}$, then

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \le \|x\|^2.$$

1.32. Def.: A Hilbert space is called <u>separable</u> if it contains a countable dense subset.

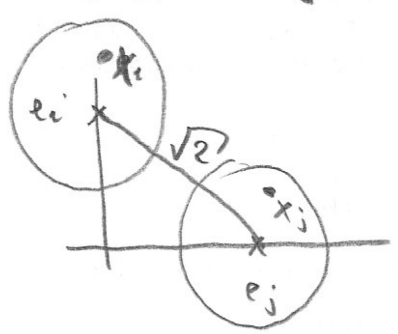
1.33. Lemma: If $\mathcal{H}$ is separable and $(e_i)_{i\in I}$ ONS in $\mathcal{H}$, then I is countable.

<u>Proof:</u> Let X be dense subset of $\mathcal{H}$

$\Rightarrow \forall i \in I \ \exists x_i \in X \ : \|x_i - e_i\| \le \frac{1}{2}$

note: $\|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j\rangle = 2$

$\Rightarrow \|e_i - e_j\| = \sqrt{2}$



$\Rightarrow x_i \neq x_j \quad \forall i \neq j$

thus: I uncountable

$\Rightarrow X$ uncountable

$\square$

1.34. Theorem: Let $(e_i)_{i\in I}$ be an ONS in a separable Hilbert space $\mathcal{H}$ (thus: I countable).
Then the following statements are equivalent:

(a) $\operatorname{span}\{e_i \mid i\in I\} := \{\sum_{\text{finite}} \alpha_i e_i \mid \alpha_i \in \mathbb{F}\}$

is dense in $\mathcal{H}$

(b) $x = \sum_{i\in I} \langle x, e_i\rangle e_i \quad \forall x\in\mathcal{H}$

(c) $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 \quad \forall x\in\mathcal{H}$

(Parseval's Identity)

Proof: (c) $\Rightarrow$ (b): Put $x_n := x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k$

to show: $x_n \to 0$

$$\|x\|^2 = \|x_n + \sum_{k=1}^{n} \langle x, e_k\rangle e_k\|^2$$

$$= \|x_n\|^2 + \underbrace{\sum_{k=1}^{n} |\langle x, e_k\rangle|^2}_{\to \|x\|^2} \quad \text{(cf. 1.29)}$$

$\Rightarrow \|x_n\| \to 0$

(b) $\Rightarrow$ (a): clear, since every $x\in\mathcal{H}$ can be approximated by finite linear combinations in (e_i)

(a) $\Rightarrow$ (c): Fix $x \in \mathcal{H}$, $\varepsilon > 0$

$\overset{a)}{\Longrightarrow} \exists M$ s.th. $\operatorname{dist}(x, \mathcal{H}_M) < \varepsilon$

where $\mathcal{H}_M = \operatorname{span}\{e_1, \dots, e_M\}$

$$x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \perp \mathcal{H}_M$$

$$\overset{1.14}{\Longrightarrow} \| x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \| = \operatorname{dist}(x, \mathcal{H}_M) < \varepsilon$$

$$\Rightarrow \underset{\substack{\uparrow \\ 1.29 \\ \text{(Bessel's Ineq.)}}}{0 \leq} \|x\|^2 - \sum_{k=1}^{M} |\langle x, e_k \rangle|^2 = \| x - \sum_{k=1}^{M} \langle x, e_k \rangle e_k \|^2 < \varepsilon^2$$

$\varepsilon \searrow 0 \Rightarrow$ (c) $\square$

1.35. Def: An ONS in a separable Hilbert space which fulfills the conditions of 1.34 is called complete ONS or orthonormal basis (ONB).

1.36. Remarks: 1) note difference between basis from linear algebra (Hamel-basis) and ONB in Hilbert space theory.

Hamel basis $\{f_i\}$: $\forall x \; \exists \alpha_i$: only finitely many $\alpha_i \neq 0$

$$x = \sum_{\text{finite}} \alpha_i f_i$$

ONB $\{e_i\}$: $\forall x \; \exists \beta_i$:

$$x = \underset{\substack{\uparrow \\ \text{possibly infinite sum}}}{\sum} \beta_i e_i$$