2) Let $(e_i)_{i\in I}$ be ONB for $\mathcal{H}$

1.34 says: • every $x \in \mathcal{H}$ can be written as

$$x = \sum_{i\in I} \alpha_i e_i \quad \text{where } \alpha_i = \langle x, e_i \rangle$$

and $\sum_{i\in I} |\alpha_i|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 = \|x\|^2 < \infty$

• this representation is unique:

$$x = \sum_{i\in I} \alpha_i e_i \implies \alpha_i = \langle x, e_i \rangle$$

3) In particular:

$(e_i)_{i\in I}$ ONB $\implies$ e_i independent

because $0 = \sum \alpha_i e_i \implies \alpha_i = 0 \quad \forall i$

~~2) $(e_i)_{i\in I}$ ONB $\Rightarrow$ e_i linearly independent:~~

~~Assume $0 = \sum \alpha_i e_i \Rightarrow \underset{\substack{\| \\ 0}}{\langle 0, e_j\rangle} = \sum \alpha_i \underbrace{\langle e_i, e_j\rangle}_{\delta_{ij}} = \alpha_j$~~

~~$\Rightarrow \alpha_j = 0 \quad \forall j$~~

1.37. Theorem (Gram-Schmidt Orthogonalization Process):

Let $(x_n)_{n=1}^N$ (where $N \in \mathbb{N} \cup \{\infty\}$) be a sequence of linearly independent elements in a Hilbert space $\mathcal{H}$. Then there is an ONS $(e_n)_{n=1}^N$ in $\mathcal{H}$ such that

$$\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{e_1, \dots, e_n\} \qquad \forall n = 1, \dots, N$$

Proof: $e_1 := \dfrac{x_1}{\|x_1\|}$

assume $e_1, \dots, e_n$ found $\rightsquigarrow$ $x'_{n+1} := x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k\rangle e_k$

$\Rightarrow x'_{n+1} \perp e_k \quad \forall\ 1 \le k \le n$

$x_{n+1} \underset{\substack{\uparrow \\ \text{linearly indep.}}}{\notin} \operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{e_1, \dots, e_n\} \Rightarrow x'_{n+1} \neq 0$

Put $e_{n+1} := \dfrac{x'_{n+1}}{\|x'_{n+1}\|}$

$\square$

1.38. Corollary: Every separable Hilbert space has a countable ONB.

Proof: $\mathcal{H}$ separable $\Rightarrow \exists$ dense sequence $(\xi_j)_{j\in\mathbb{N}}$

Cancel linearly dependent elements

$\Rightarrow \exists$ linearly independent sequence $(x_n)_{n=1}^{N}$

$(N\in\mathbb{N}$ or $N=\infty)$ s.th. $\operatorname{span}\{x_1,x_2,\dots\}$ dense in $\mathcal{H}$

$\overset{G.S.}{\Rightarrow} \exists$ ONS $(e_n)_{n=1}^{N}$ with

$\operatorname{span}\{e_1,\dots,e_n\} = \operatorname{span}\{x_1,\dots,x_n\} \quad \forall n$

$\Rightarrow \operatorname{span}\{e_1,e_2,\dots\} = \operatorname{span}\{x_1,x_2,\dots\}$ dense in $\mathcal{H}$

$\overset{1.34+1.35}{\Rightarrow} \quad (e_n)_{n=1}^{N}$ ONB $\square$

1.39. Prop.: Let $\mathcal{H}$ be a separable Hilbert space and $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ two ONB. Then I and J have the same cardinality, i.e.

$|I|=|J|\in\mathbb{N}$ or $|I|=|J|=\infty$ $(=\aleph_0=$ countable infinity$)$

Proof: Assume $|J|<\infty$

$\Rightarrow e_i = \sum_{j=1}^{|J|} \alpha_{ij} f_j$

$\overset{\text{linear algebra}}{\Rightarrow} \quad |I|\leq|J|$

$\overset{\text{symm.}}{\Rightarrow} \quad |I|=|J|$

Assume $|I|, |J|$ infinite

$\overset{1.38}{\Longrightarrow}$ $|I| =$ countably infinite $= |J|$ $\square$

<u>1.40. Definition:</u> The cardinality of an ONB of $\mathcal{H}$ is called <u>dimension of $\mathcal{H}$</u>.

Notation: $\dim \mathcal{H}$

<u>1.41. Examples:</u> 1) $\mathcal{H} = \mathbb{F}^n = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \mathbb{F}\}$

$(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ ONB

$\dim(\mathbb{F}^n) = n$

2) $\mathcal{H} = l_2 = \{(\alpha_i)_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty\}$

$e_i := (0, \dots, 0, 1, 0, \dots)$
↑ i-th position

$\Rightarrow (e_i)_{i=1}^{\infty}$ ONB $\left((\alpha_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \alpha_i e_i \right)$

$\Rightarrow \dim(l_2) = \infty$

<u>1.42. Def.:</u> Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbertspaces.

An <u>isomorphism</u> between $\mathcal{H}$ and $\mathcal{K}$ is a linear mapping $\mathcal{U}: \mathcal{H} \to \mathcal{K}$ s.th.

i) $\mathcal{U}$ surjectiv $\qquad$ $\mathcal{U}$ isometric, i.e

ii) $\langle \mathcal{U}x, \mathcal{U}y \rangle_{\mathcal{K}} = \langle x, y \rangle_{\mathcal{H}} \quad \forall x, y \in \mathcal{H}$.

$\mathcal{H}$ and $\mathcal{K}$ are <u>isomorphic</u> if such an isomorphism exists.

1.43. Remarks: 1) Assume $\mathcal{U}x = \mathcal{U}y$

$\Rightarrow \mathcal{U}(x-y) = 0$

$\Rightarrow 0 = \langle \mathcal{U}(x-y), \mathcal{U}(x-y) \rangle$

$= \langle x-y, x-y \rangle$

$= \| x-y \|^2$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\Rightarrow \mathcal{U}$ injective

thus: isomorphism is bijective

$\Rightarrow \mathcal{U}^{-1} : \mathcal{H} \to \mathcal{H}$ exists

It is an isomorphism, too:

$x = \mathcal{U}^{-1} v$, $y = \mathcal{U}^{-1} w$

$$\Rightarrow \langle \mathcal{U}x, \mathcal{U}y \rangle = \langle x, y \rangle = \langle \mathcal{U}^{-1}v, \mathcal{U}^{-1}w \rangle$$

$$\begin{array}{c} \langle \mathcal{U}x, \mathcal{U}y \rangle \\ \| \\ \langle \mathcal{U}\mathcal{U}^{-1}v, \mathcal{U}\mathcal{U}^{-1}w \rangle \\ \| \\ \langle v, w \rangle \end{array}$$

2) Note: if $\underbrace{\dim < \infty}$ and $\mathcal{H} = \mathcal{H}$, then every isometry is automatically surjective.

Not true for $\dim = \infty$

<u>1.44. Example</u>: $S: \ell_2 \to \ell_2$ one-sided shift

$$(\alpha_1, \alpha_2, \dots) \mapsto (0, \alpha_1, \alpha_2, \dots)$$

S is isometry: $\langle Sx, Sy \rangle = \langle x, y \rangle \quad \forall x, y \in \ell_2$

but not surjective: there is no $x \in \ell_2$ s.th.

$$Sx = (1, 0, 0, \dots)$$

(compare "Hilbert's hotel")

<u>1.45. Theorem</u>: Two separable Hilbert spaces are isomorphic if and only if they have the same dimension.

<u>Proof</u>: "$\Rightarrow$": Let $\mathcal{U}: \mathcal{H}_1 \to \mathcal{H}_2$ be isomorphism

If $(e_i)_{i \in I}$ is ONB of $\mathcal{H}_1$

$\Rightarrow (\mathcal{U}e_i)_{i \in I}$ ONB of $\mathcal{H}_2$

$\Rightarrow \dim \mathcal{H}_2 = |I| = \dim \mathcal{H}_1$

"$\Leftarrow$": Assume $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = |I|$

$\Rightarrow \exists$ ONB $(e_i)_{i \in I}$ of $\mathcal{H}_1$

$\exists$ ONB $(f_i)_{i \in I}$ of $\mathcal{H}_2$

Def. $\mathcal{U}: \mathcal{H}_1 \to \mathcal{H}_2$ by

$$\mathcal{U}\Big(\sum_{i \in I} \alpha_i e_i\Big) = \sum_{i \in I} \alpha_i f_i$$

$\uparrow$ every $x \in \mathcal{H}_1$ uniquely representable in this form, cpmp. 1.36.

$\mathcal{U}$ is linear, surjective and isometry:

$$\langle \mathcal{U}(\sum_i \alpha_i e_i), \mathcal{U}(\sum \beta_j e_j)\rangle = \sum_{i,j} \langle \alpha_i f_i, \beta_j f_j \rangle$$

$$= \sum_{i,j} \alpha_i \overline{\beta_j} \underbrace{\langle f_i, f_j \rangle}_{\delta_{ij}}$$

$$= \sum \alpha_i \overline{\beta_i}$$

$$= \langle \sum_i \alpha_i e_i, \sum_j \beta_j e_j \rangle \qquad \square$$

<u>1.46. Corollary</u>: A separable Hilbert space is isomorphic to $\mathbb{F}^n$ for some $n \in \mathbb{N}$ (finite dimensional case) or to $\mathbb{F}^\infty := \ell_2$ (infinite dimensional case).