11. Spectral Theory of Compact Operators

11.1. Motivation: Consider the

(Fredholm) integral equation

$$\int k(s,t)\, f(t)\, dt - \lambda f(s) = g(s),$$

where (for fixed k)

- g given
- λ parameter $\in \mathbb{C}$ ($\hat{=}$ boundary conditions)
  "eigenvalue" of the problem
- f wanted

Problem: Existence and uniqueness of solution f, depending on g and λ

abstractly: $Kf - \lambda f = g$

$$(K - \lambda 1) f = g$$

$$f = (K - \lambda 1)^{-1} g$$

thus: - when does $(K - \lambda 1)^{-1}$ exist?
- what happens if $(K - \lambda 1)^{-1}$ does not exist?

Consider finite-dimensional case:

A $n \times n$-matrix, then

$(A - \lambda 1)^{-1}$ does not exist

$\Leftrightarrow (A - \lambda 1) : \mathbb{C}^n \to \mathbb{C}^n$ not bijective

$\overset{\text{finite}}{\underset{\text{dimensions}}{\Leftrightarrow}} (A - \lambda 1)$ not injective

$\Leftrightarrow \exists\, x \neq 0 : (A - \lambda 1) x = 0$

$\Leftrightarrow \lambda$ eigenvalue $: \exists\, x \neq 0 : Ax = \lambda x$

<u>11.2. Def.</u>: Let X be a (complex) Banach space and $A \in B(X)$

1) The <u>spectrum</u> of A is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda \cdot 1) : X \to X \text{ not bijective}\}$$

$$\subset \mathbb{C}$$

($1 \hat{=}$ id identity map)

The complement

$$\rho(A) := \mathbb{C} \setminus \sigma(A)$$

is called <u>resolvent set</u> of A.

2) $\lambda \in \mathbb{C}$ is an eigenvalue of A, if

$$\exists \underset{x \neq 0}{x \in X} : Ax = \lambda x$$

(i.e., $A - \lambda 1$ is not injective)

x is then an eigenvector for λ.

3) $\sigma_p(A) := \{\lambda \in \mathbb{C} \mid \lambda \text{ eigenvalue of } A\}$

is the <u>point spectrum</u> of A.

<u>11.3. Remarks:</u> 1) We have $\sigma_p \subset \sigma$

If $\dim X < \infty$, then $\sigma = \sigma_p$,

but for $\dim X = \infty$ we have in general

$$\sigma_p \subsetneq \sigma$$

2) $\sigma_p = \emptyset$ can happen but, as we will see later, $\sigma \neq \emptyset$ always

3) $\lambda \notin \sigma(A) \Rightarrow (A - \lambda 1)$ bijective

$$\overset{7.4.}{\Longrightarrow} (A - \lambda 1)^{-1} \in B(X)$$

and thus

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda 1)^{-1} \in B(X)$$

<u>11.4. Example:</u> Consider $X = C[1,2]$ and $A : X \to X$ with

$$(Af)(t) = t\, f(t)$$

Then $A \in \mathcal{B}(X)$ (with $\|A\| = 2$) and

i) $\sigma_p(A) = \emptyset$

indeed: $Af = \lambda f$ means

$$t\, f(t) = \lambda f(t) \quad \forall\, 1 \le t \le 2$$

$$\Rightarrow f(t) = 0 \quad \forall\, t \neq \lambda$$

$$\Rightarrow f = 0$$

ii) $\sigma(A) = [1,2]$

indeed: $\lambda \notin [1,2] \Rightarrow (A - \lambda \mathbb{1})^{-1}$ is given by

$$[(A - \lambda \mathbb{1})^{-1} f](t) = \frac{1}{t - \lambda} f(t)$$

$\lambda \in [1,2]$: $(A - \lambda \mathbb{1})$ is not surjective, since $(A - \lambda \mathbb{1}) f(t) = (t - \lambda) f(t)$ has for each f a zero at $t = \lambda$; and thus $g(t) = 1$ $(1 \le t \le 2)$ is not in $\operatorname{ran}(A - \lambda \mathbb{1})$

Such phenomena are typical for $A \in B(X)$ with $\dim X = \infty$. For compact operators, however, they do not occur; there everything is more or less like in finite dimensions. We will show the following

<u>11.5. Theorem:</u> Let X be a Banach space and $T \in \mathcal{K}(X)$. Then we have:

i) $\sigma(T)$ is a finite or countable compact subset of $\mathbb{C}$, the only possible cluster point is 0.

If X is infinite-dimensional, then $0 \in \sigma(T)$ (i.e. T is not bijective)

ii) Each $\lambda \in \sigma(T) \setminus \{0\}$ is eigenvalue of T and $\dim \ker(T - \lambda 1) < \infty$

iii) For each $\lambda \in \mathbb{C} \setminus \{0\}$ we have the Fredholm alternative:

$T - \lambda 1$ surjective $\Leftrightarrow$ $T - \lambda 1$ injective, i.e for $\lambda \neq 0$ we have:

$$\left.\begin{array}{l}\text{The inhomogeneous}\\ \text{equation}\\ (T-\lambda)x = y\\ \text{has for each } y\\ \text{exactly one solution}\end{array}\right\} \quad \left\{\begin{array}{l}\text{The homogeneous}\\ \text{equation}\\ (T-\lambda)x = 0\\ \text{has as only}\\ \text{solution } x = 0\end{array}\right.$$

existence $\Longleftrightarrow$ uniqueness

<u>11.6 Remark</u>: This spectral theorem makes statements about

- T itself (for $\lambda = 0$)
- $(T - \lambda 1) = -\lambda(1 - \frac{1}{\lambda}T)$ for $\lambda \neq 0$

Since T compact $\Leftrightarrow \frac{1}{\lambda}T$ compact the case $\lambda \neq 0$ corresponds to the investigation of

$$A = 1 - T \quad \text{if } T \text{ compact}$$

Idea: A is a compact (small) deformation of 1, thus properties of A should be close to properties of 1.

11.7. Proposition: Let X be a Banach space and $T \in \mathcal{K}(X)$. Put $A := 1 - T$. Then we have

i) $\ker A$ is finite-dimensional

ii) $\operatorname{ran} A$ is closed in X

iii) $\operatorname{ran} A$ has finite codimension in X

(i.e. $\exists$ finitely many $x_1, \dots, x_n \in X$ s.th. $\operatorname{span}(\operatorname{ran} A, x_1, \dots, x_n) = X$)

Proof: i) $\ker A \subset X$ closed linear subspace, i.e. Banach space

$$B_1(\ker A) = \{\underbrace{x \in \ker A}_{x = Tx} \mid \|x\| \leq 1\} \subset \overline{T(B_1(X))}$$

$\uparrow$ compact set, since $T \in \mathcal{K}(X)$

$\Rightarrow B_1(\ker A)$ compact

$\overset{8.2}{\Rightarrow} \ker A$ is finite-dimensional

ii) Fix $y \in \overline{\operatorname{ran} A}$, i.e. $\exists\, y_n = Ax_n$ s.th. $y_n \to y$

to show: $y \in \operatorname{ran} A$, i.e. $\exists x \in X : y = Ax$

(ii) Fix $y \in \overline{\operatorname{ran} A}$, i.e. $\exists\, y_n = A x_n$ s.th. $y_n \to y$

to show: $\exists\, x \in X$ s.th. $y = Ax$

we need: (x_n) bounded (for compactness argument)

but note: $A x_n = A \tilde{x}_n$ for $x_n - \tilde{x}_n \in \ker A$,

thus we can only hope to keep $(\operatorname{dist}(x_n, \ker A))_{n \in \mathbb{N}}$ bounded

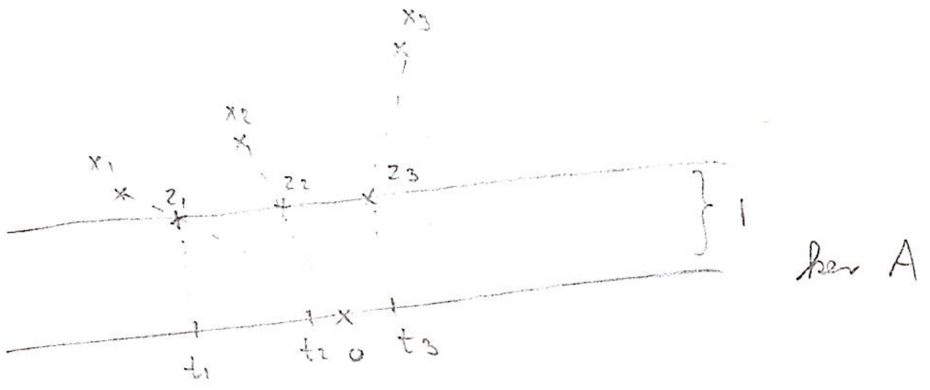
We show this by contradiction

Assume without restriction (passing to subsequence) that

$$\lim_{n \to \infty} \operatorname{dist}(x_n, \ker A) = \infty$$

Put $z_n := \dfrac{x_n}{\operatorname{dist}(x_n, \ker A)} \Rightarrow \operatorname{dist}(z_n, \ker A) = 1$

$\Rightarrow \exists\, t_n \in \ker A$ s.th. $\|z_n - t_n\| \leq 2$



Put $s_n := z_n - t_n \Rightarrow \|s_n\| \leq 2$

$\operatorname{dist}(s_n, \ker A) = \operatorname{dist}(z_n, \ker A) = 1$

$$A s_n = A z_n = \frac{A x_n}{\operatorname{dist}(x_n, \ker A)} \longrightarrow \frac{y}{\infty} = 0$$

$(s_n)_{n\in\mathbb{N}}$ bounded $\xRightarrow{T\text{ compact}}$ $\exists$ subsequence s.th.

$(T s_{n(j)})_j$ converges

i.e. $\exists\, a \in X$: $T s_{n(j)} \xrightarrow{j\to\infty} a$

$$\begin{array}{ccccc} T s_{n(j)} = (I - A)\, s_{n(j)} = s_{n(j)} - & A s_{n(j)} \\ \downarrow & \downarrow \\ a & 0 \end{array}$$

$\Rightarrow$ $s_{n(j)} \to a$

$\Rightarrow$ $\operatorname{dist}(a, \ker A) = \lim\limits_{j\to\infty} \operatorname{dist}(s_{n(j)}, \ker A) = 1$

$A a = \lim\limits_{j\to\infty} A s_{n(j)} = 0$ contradiction

thus: $\exists M < \infty$: $\operatorname{dist}(x_n, \ker A) \le M$ $\forall n$

Let $t_n \in \ker A$ be such that $\|x - t_n\| \le M + 1$

Put $\tilde{x}_n := x - t_n$

$\Rightarrow$ $A\tilde{x}_n = A x_n$ and $\|\tilde{x}_n\| \le M+1$

T compact $\Rightarrow$ $\exists$ convergent subsequence $(T\tilde{x}_{n(j)})_j$,

i.e. $\exists\, b \in X$ s.th. $T\tilde{x}_{n(j)} \to b$

$$\begin{array}{c} T\tilde{x}_{n(j)} = (I - A)\,\tilde{x}_{n(j)} = \tilde{x}_{n(j)} - \underbrace{A\tilde{x}_{n(j)}}_{A x_{n(j)} \to y} \\ \downarrow \\ b \end{array}$$

$\Rightarrow$ $\tilde{x}_{n(j)} \to b + y$

$\Rightarrow \quad A\tilde{x}_{n(j)} \to A(b+y)$

$$\begin{array}{c} A\tilde{x}_{n(j)} \\ \downarrow \\ y \end{array}$$

$\Rightarrow \quad y = A(b+y)$

i.e. $y \in \operatorname{ran} A$

(iii) Assume $\operatorname{codim} A = \infty$, i.e. $\exists\, x_i \in X \ (i \in \mathbb{N})$ s.th.

$$x_{n+1} \notin \underbrace{\operatorname{span}(\operatorname{ran} A, x_1, \dots, x_n)}_{=: V_n \subset X \text{ linear subspace}}$$

By 8.1, we can choose x_i in such a way that

$\|x_n\| = 1$ and $\operatorname{dist}(x_n, V_{n-1}) \geq \frac{1}{2}$

$$\Rightarrow \|Tx_n - Tx_m\| = \|x_n \underbrace{- Ax_n - x_m + Ax_m}_{\in V_{n-1} \quad \text{for } n > m}\|$$

$$\geq \operatorname{dist}(x_n, V_{n-1})$$

$$\geq \frac{1}{2}$$

$\Rightarrow$ $(Tx_n)_n$ has no convergent subsequence, although $(x_n)_n$ is bounded

contradiction to T compact

$\Rightarrow \quad \operatorname{codim} \operatorname{ran} A < \infty$ $\square$

11.8. Theorem: Let X be a Banach space and $T \in \mathcal{K}(X)$. Put $A := I - T$. Then we have:

(i) $(\ker A^n)_{n\geq 1}$ is a monotone increasing sequence of finite-dimensional subspaces of X and there is an $n \geq 1$ s. th.

$$\ker A^n = \ker A^{n+1} =: N$$

(ii) $(\operatorname{ran} A^m)_{m\geq 1}$ is a monotone decreasing sequence of closed subspaces with finite codimension and we have (with n from (i)):

$$\operatorname{ran} A^n = \operatorname{ran} A^{n+1} =: R$$

(iii) We have: $X = N \oplus R$ and

$A : R \to R$ is isomorphism

11.9. Example: In finite dimensions this corresponds to Jordan form

$$T = \begin{pmatrix} 1 & 1 & 0 & & \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 1 & & \\ & 0 & & \lambda & 1 \\ & & & 0 & \lambda \end{pmatrix}$$

$\lambda_1 = 1$

$\lambda_2 = \lambda \neq 1$

$$\Rightarrow A = I - T = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \\ & 0 & & \tilde{\lambda} & 1 \\ & & & 0 & \tilde{\lambda} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} N \\ \left.\vphantom{\begin{matrix}0\\0\end{matrix}}\right\} R \end{matrix}$$

$\tilde{\lambda} = 1 - \lambda \neq 0$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ & & & \tilde\lambda^2 & * \\ & 0 & & 0 & \tilde\lambda^2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ & & & \tilde\lambda^3 & * \\ & 0 & & 0 & \tilde\lambda^3 \end{pmatrix}$$

$$\Rightarrow \ker A = \mathbb{C} \oplus 0 \oplus 0 \oplus 0 \oplus 0$$

$$\ker A^2 = \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus 0 \oplus 0$$

$$\ker A^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus 0 = \ker A^4 =: N$$

$$\operatorname{ran} A = \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}$$

$$\operatorname{ran} A^2 = \mathbb{C} \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}$$

$$\operatorname{ran} A^3 = 0 \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C} = \operatorname{ran} A^4 =: R$$

$$\mathbb{C}^5 = N \oplus R$$

$A : R \to R$ Isomorphism

<u>Proof:</u> note: $A^n = (I - T)^n = I - T_n$

where T_n compact

(by induction: $A^n = (I-T)\underbrace{(I-T)^{n-1}}_{I - T_{n-1}}$

$$= I - \underbrace{T_{n-1} - T + T \cdot T_{n-1}}_{=: T_n \text{ compact}}$$)

thus we have statements from 10.2. also for A^n, i.e.

- $\ker A^n$ finite-dimensional
- $\operatorname{ran} A^n$ closed and of finite codimension

monotonicity of sequences:

$\ker A^n \subset \ker A^{n+1}$: $A^n x = 0 \Rightarrow A^{n+1} x = A(A^n x) = 0$

$\operatorname{ran} A^n \supset \operatorname{ran} A^{n+1}$: $y = A^{n+1} x = A^n (Ax)$

to show: $\exists n$ s.th. $\ker A^n = \ker A^{n+1}$

Assume $\ker A^n \neq \ker A^{n+1}$ $\forall n$

$\Rightarrow \exists$ sequence $(x_n)_n$ s.th. $x_n \in \ker A^n$, but $x_n \notin \ker A^{n-1}$

By 8.1, we can choose x_n in such a way that

$\|x_n\| = 1$ and $\operatorname{dist}(x_n, \ker A^{n-1}) \geq \frac{1}{2}$

$(x_n)_n$ bounded $\overset{T \text{ compact}}{\Longrightarrow}$ (Tx_n) has convergent subsequence

but: $\|Tx_n - Tx_m\| = \| x_n \underbrace{- Ax_n - x_m + Ax_m}_{\in \ker A^{n-1} \quad (\text{if } n > m)} \|$

$\geq \operatorname{dist}(x_n, \ker A^{n-1})$

$\geq \frac{1}{2}$ contradiction

$\Rightarrow \exists n : \ker A^n = \ker A^{n+1} = \ker A^{n+2} = \ldots$

$(A^{n+2} x = 0 \Rightarrow A^{n+1}(Ax) = 0 \Rightarrow Ax \in \ker A^{n+1} = \ker A^n$

$\Rightarrow A^n A x = 0$

in the same way for $\operatorname{ran} A^m$: Assume $\operatorname{ran} A^m \neq \operatorname{ran} A^{m+1}$ $\forall$

$\Rightarrow \exists$ sequence $(x_m)_m$ s.th. $x_m \in \operatorname{ran} A^m$, $x_m \notin \operatorname{ran} A^{m+1}$

$\|x_m\| = 1$, $\operatorname{dist}(x_m, A^{m+1}) \geq \frac{1}{2}$

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$(x_m)_m$ bounded $\overset{T\text{ compact}}{\Longrightarrow}$ $\exists$ convergent subsequence of (Tx_n)

but: $\|Tx_n - Tx_m\| = \| \underbrace{x_n - Ax_n + Ax_m}_{\in \operatorname{ran} A^{m+1}} - x_m \|$ (if $n > m$)

$\geq \operatorname{dist}(x_m, A^{m+1})$

$\geq \frac{1}{2}$ contradiction

$\Rightarrow \exists m : \operatorname{ran} A^m = \operatorname{ran} A^{m+1} = \operatorname{ran} A^{m+2} = \dots$

Let n_0, m_0 be the smallest n, m with above properties.

We will show: $n_0 \geq m_0$

first observe: $\ker A^{n_0} \cap \operatorname{ran} A^{n_0} = \{0\}$

(because: $y \in \ker A^{n_0} \cap \operatorname{ran} A^{n_0}$

$\Rightarrow y = A^{n_0} x \Rightarrow 0 = A^{n_0} y = A^{2n_0} x$

$\Rightarrow x \in \ker A^{2n_0} = \ker A^{n_0}$

$\Rightarrow y = A^{n_0} x = 0$)

Assume $n_0 < m_0$, i.e.

$\operatorname{ran} A^{m_0} \underset{\neq}{\subset} \operatorname{ran} A^{m_0-1} \subset \operatorname{ran} A^{n_0}$ ($n_0 \leq m_0 - 1$)

$\Rightarrow \exists z \notin \operatorname{ran} A^{m_0}$ s.th. $z \in \operatorname{ran} A^{m_0-1} \subset \operatorname{ran} A^{n_0}$

$\Downarrow$

$Az \in \operatorname{ran} A^{m_0} = \operatorname{ran} A^{m_0+1}$

$\Rightarrow Az = Ax$ where $x \in \operatorname{ran} A^{m_0} \subset \operatorname{ran} A$

$\Rightarrow A(z-x) = 0$

i.e. $z - x \in \ker A \subset \ker A^{n_0}$

but: $z, x \in \operatorname{ran} A^{n_0}$

$\Rightarrow z - x \in \operatorname{ran} A^{n_0} \cap \ker A^{n_0} = \{0\}$

$\Rightarrow z = x \in \operatorname{ran} A^{m_0}$ contradiction

$\Rightarrow n_0 \geq m_0$

(iii) $N := \ker A^{n_0}$

$R := \operatorname{ran} A^{m_0} = \operatorname{ran} A^{n_0}$

we have shown above: $N \cap R = \{0\}$

to show: $N + R = X$

Consider $x \in X$

(want to write it as $x = \tilde{y} + y$, $\tilde{y} \in \ker A^{n_0}$, $y \in \operatorname{ran} A^{n_0}$

$\Rightarrow A^{n_0} x = A^{n_0} y$)

$A^{n_0} x \in \operatorname{ran} A^{n_0} = \operatorname{ran} A^{2n_0}$, i.e. $A^{n_0} x = A^{n_0} y$

where $y \in \operatorname{ran} A^{n_0} = R$

$\Rightarrow A^{n_0}(x - y) = 0$

$\Rightarrow x - y \in \ker A^{n_0} = N$

$$\Rightarrow x = \underbrace{(x-y)}_{\substack{\cap \\ N}} + \underset{\substack{\cap \\ R}}{y}$$

$\Rightarrow X = N \oplus R$

remains to show: $A: R \to R$ isomorphism

note: $A|R$ is surjective, since
$$A(R) = A(\operatorname{ran} A^{n_0}) = \operatorname{ran} A^{n_0+1} = \operatorname{ran} A^{n_0} = R$$

and injective, since
$$\ker A|_R \subset \underbrace{\ker A}_{\subset \ker A^{n_0} = N} \cap R \subset N \cap R = \{0\}$$

thus: $A: R \to R$ bijective

$$\left.\begin{array}{l} A \text{ bounded} \\ R \text{ Banach space} \\ \text{(since closed)} \end{array}\right\} \overset{7.5.}{\Longrightarrow} A: R \to R \text{ isomorphism}$$

$\square$

Proof of Theorem 11.5 (ii) Consider $\lambda \in \sigma(T) \setminus \{0\}$

$$\Rightarrow (T - \lambda I) = \underbrace{-\lambda (I - \underbrace{\tfrac{1}{\lambda} T}_{\text{compact}})}_{=: A_\lambda}$$

$\overset{11.8}{\Longrightarrow}$ $X = N_\lambda \oplus R_\lambda$ s.th.

$A_\lambda : R_\lambda \to R_\lambda$ isomorphism

and

$N_\lambda = \ker A_\lambda^{n_0}$,

$\ker A_\lambda^{n_0 - 1} \underset{\neq}{\subseteq} \ker A_\lambda^{n_0} = \ker A_\lambda^{n_0+1}$

$\uparrow$ if $n_0 \geq 1$

$\lambda \in \sigma(T) \Rightarrow N_\lambda \neq \{0\}$ (otherwise: $R_\lambda = X$ and
(thus $n_0 \geq 1$) $A_\lambda : R_\lambda \to R_\lambda$ isom.)

$\Rightarrow \exists\, x \in \ker A_\lambda^{n_0} \setminus \ker A_\lambda^{n_0 - 1}$

i.e. $y := A_\lambda^{n_0 - 1} x \neq 0$ and $A_\lambda y = A_\lambda^{n_0} x = 0$

$\Rightarrow \ker A_\lambda \neq \{0\}$

$\quad \|$

$\ker (T - \lambda I)$ $\Rightarrow \lambda \in \sigma_p(T)$

furthermore: $\ker(T - \lambda I) = \ker A_\lambda \subseteq \ker A_\lambda^{n_0} = N_\lambda$

and $\dim N_\lambda < \infty$. (by 10.8)

$\Rightarrow \dim \ker (T - \lambda I) < \infty$

(iii) Consider $\lambda \in \mathbb{C} \setminus \{0\}$

$T - \lambda I$ injective $\Rightarrow \lambda \notin \sigma_p(T)$

$\overset{(ii)}{\Rightarrow} \lambda \notin \sigma(T)$

$\Rightarrow T - \lambda I$ bijective, in particular surjective

$T - \lambda I$ surjective $\Rightarrow A_\lambda = I - \frac{1}{\lambda} T$ surjective

$\Rightarrow A_\lambda^m$ surjective $\forall m = 1, 2, \ldots$

$\Rightarrow R = \operatorname{ran} A_\lambda^{m_0} = X$

$$\Big\Downarrow \; A_\lambda : \underset{X}{\overset{R}{\|}} \to \underset{X}{\overset{R}{\|}} \quad \text{isom}$$

A_λ injective

$\Rightarrow T - \lambda I$ injective

(i) We will show: Each $\lambda \in \sigma(T) \setminus \{0\}$ is isolated in $\sigma(T)$

i.e. $\exists \, \varepsilon > 0 : \; \mathcal{U}_\varepsilon(\lambda) \cap \sigma(T) = \{\lambda\}$

i.e. we have to show:

$T - (\lambda + \mu) I$ is invertible for $0 < |\mu| \leq \varepsilon$

We have: $T - (\lambda+\mu) I = \underbrace{(T - \lambda I)}_{\text{we know this}} - \mu I$

$X = N_\lambda \oplus R_\lambda$ and

$T - \lambda I : R_\lambda \to R_\lambda$ isomorphism

and

$\exists n : (T-\lambda I)^n \equiv 0$ on N_λ

note: $(T-\lambda I) - \mu I : R_\lambda \to R_\lambda$
$N_\lambda \to N_\lambda$

hence, it suffices to show that

$(T-(\lambda+\mu)I)|R_\lambda$ and $(T-(\lambda+\mu)I)|N_\lambda$

are both invertible.

This follows from

11.10. Lemma: Let X be a Banach space and $A \in B(X)$:

1) If A is nilpotent (i.e. $\exists n : A^n = 0$), then

$\sigma(A) = \{0\}$

(i.e. $A - \mu I$ is invertible $\forall \mu \neq 0$)

2) If $\lambda \notin \sigma(A)$, then $\exists \varepsilon > 0$ s.th.

$\lambda + \mu \notin \sigma(A) \quad \forall |\mu| \leq \varepsilon$

(i.e. $\rho(A)$ is open, i.e. $\sigma(A)$ is closed)

<u>Proof:</u> note: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ if this makes sense

1) $(A-\mu I)^{-1} = -\mu^{-1}(I-\mu^{-1}A)^{-1}$ $(\mu \neq 0)$

$$= -\mu^{-1} \sum_{k=0}^{\infty} \frac{A^k}{\mu^k}$$

$$= -\mu^{-1} \sum_{k=0}^{n-1} \frac{A^k}{\mu^k} \quad \text{is inverse}$$

2) $\lambda \notin \sigma(A) \Rightarrow (A-\lambda I)^{-1}$ exists

$$A-(\lambda+\mu)I = (A-\lambda I)\,[I-\mu(A-\lambda I)^{-1}]$$

$$\Rightarrow (A-(\lambda+\mu)I)^{-1} = [\;]^{-1}(A-\lambda I)^{-1}$$

if $[\;]^{-1}$ exists

Put $B := \mu(A-\lambda I)^{-1}$

$|\mu|$ suff. small $\Rightarrow \|B\| < 1$

$$\Rightarrow (I-B)^{-1} = \underbrace{\sum_{n=0}^{\infty} B^n}_{\text{converges, since } \|B\|<1}$$

thus: $|\mu|$ suff. small $\Rightarrow (A-(\lambda+\mu)I)^{-1} \in B(X)$ exist

$\Rightarrow \lambda+\mu \notin \sigma(A)$

$\square$

end of proof of ~~10.12~~ 11.5

(i) 11.10 $\Rightarrow$ all $\lambda \in \sigma(T) \setminus \{0\}$ are isolated in $\sigma(T)$

$\Rightarrow$ only possible limit point is 0

$\Rightarrow$ $\sigma(T)$ is countable (finite or countably infinite)

Assume $0 \notin \sigma(T)$, i.e. T bijective

$\xRightarrow[\text{Mapp. Th.}]{\text{Inverse}}$ $T^{-1} \in B(X)$

$\Rightarrow$ $I = T\,T^{-1} \in \mathcal{K}(X)$, since $\mathcal{K}(X)$ ideal

$\Rightarrow$ X finite-dimensional

thus, for $\dim X = \infty$: $\sigma(T)$ compact, since

closed $(0 \in \sigma(T))$

bounded $(\sigma(T) \subset \{\lambda \mid |\lambda| \leq \|T\|$

$\uparrow$

$\lambda \in \sigma_p(T) \Rightarrow |\lambda| \leq \|T\|$

for $\dim X < \infty$: $\sigma(T)$ compact, too, since finite set

$\square$