# 11. Spectral Theory of Compact Operators

**11.1. Motivation:** Consider the (Fredholm) integral equation

$$\int k(s,t)\, f(t)\, dt - \lambda f(s) = g(s),$$

where (for fixed $k$)

- $g$ given
- $\lambda$ parameter $\in \mathbb{C}$ ($\hat{=}$ boundary condition) "eigenvalue" of the problem
- $f$ wanted

Problem: Existence and uniqueness of solution $f$, depending on $g$ and $\lambda$

abstractly:
$$K f - \lambda f = g$$
$$(K - \lambda 1) f = g$$
$$f = (K - \lambda 1)^{-1} g$$

thus:
- when does $(K - \lambda 1)^{-1}$ exist?
- what happens if $(K - \lambda 1)^{-1}$ does not exist?

Consider finite-dimensional case:

$A$ $n \times n$ - matrix, then

$(A - \lambda 1)^{-1}$ does not exist

$\Leftrightarrow (A - \lambda 1): \mathbb{C}^n \to \mathbb{C}^n$ not bijective

$\overset{\text{finite}}{\underset{\text{dimensions}}{\Leftrightarrow}} (A - \lambda 1)$ not injective

$\Leftrightarrow \exists x \neq 0 : (A - \lambda 1) x = 0$

$\Leftrightarrow \lambda$ eigenvalue : $\exists x \neq 0 : Ax = \lambda x$

<u>11.2. Def.</u>: Let $X$ be a (complex) Banach space and $A \in B(X)$

1) The <u>spectrum</u> of $A$ is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda \cdot 1) : X \to X \text{ not bijective}\}$$

$$\subset \mathbb{C}$$

($1 \hat{=}$ id identity map)

The complement

$$\rho(A) := \mathbb{C} \setminus \sigma(A)$$

is called <u>resolvent set</u> of $A$.

2) $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, if

$$\exists \underset{x \neq 0}{x \in X} : Ax = \lambda x$$

(i.e., $A - \lambda 1$ is not injective)

$x$ is then an eigenvector for $\lambda$.

3) $\sigma_p(A) := \{\lambda \in \mathbb{C} \mid \lambda \text{ eigenvalue of } A\}$

is the <u>point spectrum</u> of $A$.

<u>11.3. Remarks:</u> 1) We have $\sigma_p \subset \sigma$

If $\dim X < \infty$, then $\sigma = \sigma_p$,

but for $\dim X = \infty$ we have in general

$$\sigma_p \subsetneqq \sigma$$

2) $\sigma_p = \emptyset$ can happen but, as we will see later, $\sigma \neq \emptyset$ always

3) $\lambda \notin \sigma(A) \Rightarrow (A - \lambda 1)$ bijective

$$\overset{7.4.}{\Longrightarrow} (A - \lambda 1)^{-1} \in B(X)$$

and thus

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda 1)^{-1} \in B(X)$$

<u>11.4. Example:</u> Consider $X = C[1,2]$

and $A : X \to X$ with

$$(Af)(t) = t\,f(t)$$

Then $A \in B(X)$ (with $\|A\| = 2$) and

i) $\sigma_p(A) = \emptyset$

indeed: $Af = \lambda f$ means

$$t\,f(t) = \lambda f(t) \quad \forall\, 1 \le t \le 2$$

$$\Rightarrow f(t) = 0 \quad \forall\, t \neq \lambda$$

$$\Rightarrow f = 0$$

ii) $\sigma(A) = [1,2]$

indeed: $\lambda \notin [1,2] \Rightarrow (A - \lambda 1)^{-1}$ is given by

$$[(A - \lambda 1)^{-1} f](t) = \frac{1}{t-\lambda} f(t)$$

$\lambda \in [1,2]$: $(A - \lambda 1)$ is not surjective, since $(A - \lambda 1) f(t) = (t - \lambda) f(t)$ has for each $f$ a zero at $t = \lambda$; and thus $g(t) = 1$ $(1 \le t \le 2)$ is not in $\operatorname{ran}(A - \lambda 1)$