Such phenomena are typical for $A \in B(X)$ with $\dim X = \infty$. For compact operators, however, they do not occur; there everything is more or less like in finite dimensions. We will show the following

<u>11.5. Theorem:</u> Let X be a Banach space and $T \in \mathcal{K}(X)$. Then we have:

i) $\sigma(T)$ is a finite or countable compact subset of $\mathbb{C}$, the only possible cluster point is 0.

If X is infinite-dimensional, then $0 \in \sigma(T)$ (i.e. T is not bijective)

ii) Each $\lambda \in \sigma(T) \setminus \{0\}$ is eigenvalue of T and $\dim \ker(T - \lambda 1) < \infty$

iii) For each $\lambda \in \mathbb{C} \setminus \{0\}$ we have the Fredholm alternative:

$T - \lambda 1$ surjective $\Leftrightarrow$ $T - \lambda 1$ injective, i.e. for $\lambda \neq 0$ we have:

The inhomogeneous equation $(T-\lambda)x = y$ has for each y exactly one solution	$\Leftrightarrow$	The homogeneous equation $(T-\lambda)x = 0$ has as only solution $x = 0$

existence $\Leftarrow$ uniqueness

11.6 Remark: This spectral theorem makes statements about

- T itself (for $\lambda = 0$)
- $(T - \lambda 1) = -\lambda\left(1 - \frac{1}{\lambda} T\right)$ for $\lambda \neq 0$

Since T compact $\Leftrightarrow$ $\frac{1}{\lambda} T$ compact the case $\lambda \neq 0$ corresponds to the investigation of

$$A = 1 - T \quad \text{if } T \text{ compact}$$

Idea: A is a compact (small) deformation of 1, thus properties of A should be close to properties of 1.

11.7. Proposition: Let X be a Banach space and $T \in \mathcal{K}(X)$. Put $A := 1 - T$.

Then we have

i) $\ker A$ is finite-dimensional

ii) $\operatorname{ran} A$ is closed in X

iii) $\operatorname{ran} A$ has finite codimension in X

(i.e. $\exists$ finitely many $x_1, \dots, x_n \in X$ s.th. $\operatorname{span}(\operatorname{ran} A, x_1, \dots, x_n) = X$)

Proof: i) $\ker A \subset X$ closed linear subspace, i.e. Banach space

$$B_1(\ker A) = \{\underbrace{x \in \ker A}_{x = Tx} \mid \|x\| \leq 1\} \subset \overline{T(B_1(X))}$$

compact set, since $T \in \mathcal{K}(X)$

$\Rightarrow B_1(\ker A)$ compact

$\overset{8.2}{\Rightarrow}$ $\ker A$ is finite-dimensional

ii) Fix $y \in \overline{\operatorname{ran} A}$, i.e. $\exists y_n = Ax_n$ s.th. $y_n \to y$

to show: $y \in \operatorname{ran} A$, i.e. $\exists x \in X : y = Ax$

(ii) Fix $y \in \overline{\operatorname{ran} A}$, i.e. $\exists\, y_n = A x_n$ s.th. $y_n \to y$

to show: $\exists\, x \in X$ s.th. $y = Ax$

we need: (x_n) bounded (for compactness argument)

but note: $A x_n = A \tilde{x}_n$ for $x_n - \tilde{x}_n \in \ker A$,

thus we can only hope to keep $(\operatorname{dist}(x_n, \ker A))_{n \in \mathbb{N}}$ bounded

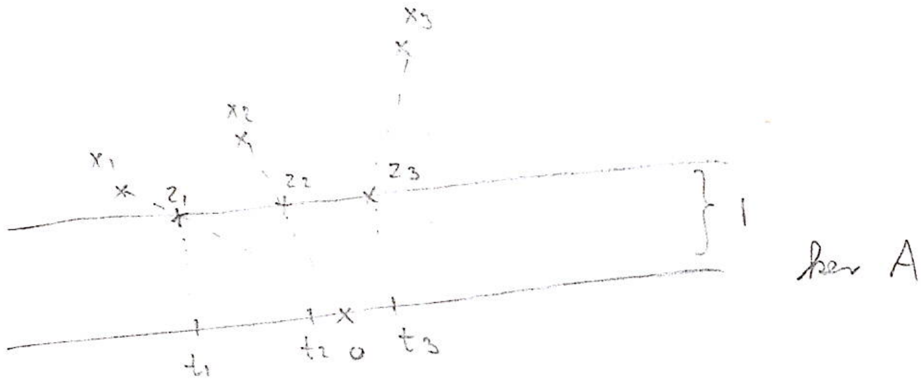
We show this by contradiction

Assume without restriction (passing to subsequence) that

$$\lim_{n \to \infty} \operatorname{dist}(x_n, \ker A) = \infty$$

Put $z_n := \dfrac{x_n}{\operatorname{dist}(x_n, \ker A)} \Rightarrow \operatorname{dist}(z_n, \ker A) = 1$

$\Rightarrow \exists\, t_n \in \ker A$ s.th. $\|z_n - t_n\| \leq 2$



Put $s_n := z_n - t_n \Rightarrow \|s_n\| \leq 2$

$\operatorname{dist}(s_n, \ker A) = \operatorname{dist}(z_n, \ker A) = 1$

$$A s_n = A z_n = \frac{A x_n}{\operatorname{dist}(x_n, \ker A)} \to \frac{y}{\infty} = 0$$

$(s_n)_{n\in\mathbb{N}}$ bounded $\xRightarrow{T \text{ compact}}$ $\exists$ subsequence s.th.

$(T s_{n(j)})_j$ converges

i.e. $\exists\, a \in X$: $T s_{n(j)} \xrightarrow{j\to\infty} a$

$$\begin{array}{ccccc} T s_{n(j)} = & (I - A)\, s_{n(j)} & = & s_{n(j)} - & A s_{n(j)} \\ \downarrow & & & & \downarrow \\ a & & & & 0 \end{array}$$

$\Rightarrow$ $s_{n(j)} \to a$

$\Rightarrow$ $\operatorname{dist}(a, \ker A) = \lim_{j\to\infty} \operatorname{dist}(s_{n(j)}, \ker A) = 1$

$A a = \lim_{j\to\infty} A s_{n(j)} = 0$ contradiction

thus: $\exists M < \infty$: $\operatorname{dist}(x_n, \ker A) \le M$ $\forall n$

Let $t_n \in \ker A$ be such that $\|x - t_n\| \le M + 1$

Put $\tilde{x}_n := x - t_n$

$\Rightarrow$ $A\tilde{x}_n = A x_n$ and $\|\tilde{x}_n\| \le M+1$

T compact $\Rightarrow$ $\exists$ convergent subsequence $(T\tilde{x}_{n(j)})_j$,

i.e. $\exists\, b \in X$ s.th. $T\tilde{x}_{n(j)} \to b$

$$\begin{array}{ccc} T\tilde{x}_{n(j)} = & (I - A)\,\tilde{x}_{n(j)} = & \tilde{x}_{n(j)} - \underbrace{A\tilde{x}_{n(j)}}_{A x_{n(j)} \to y} \\ \downarrow & & \\ b & & \end{array}$$

$\Rightarrow$ $\tilde{x}_{n(j)} \to b + y$

$$\Rightarrow \quad \begin{array}{ccc} A\tilde{x}_{n(j)} & \longrightarrow & A(b+y) \\ \downarrow & & \\ y & & \end{array}$$

$\Rightarrow \quad y = A(b+y)$

i.e. $y \in \operatorname{ran} A$

(iii) Assume $\operatorname{codim} A = \infty$, i.e. $\exists\, x_i \in X$ $(i \in \mathbb{N})$ s.th.

$$x_{n+1} \notin \underbrace{\operatorname{span}(\operatorname{ran} A, x_1, \dots, x_n)}_{=:\, V_n \subset X \text{ linear subspace}}$$

By 8.1, we can choose x_i in such a way that

$\|x_n\| = 1$ and $\operatorname{dist}(x_n, V_{n-1}) \geq \frac{1}{2}$

$$\Rightarrow \|Tx_n - Tx_m\| = \|x_n \underbrace{- Ax_n - x_m + Ax_m}_{\in V_{n-1} \quad \text{for } n > m}\|$$

$$\geq \operatorname{dist}(x_n, V_{n-1})$$

$$\geq \frac{1}{2}$$

$\Rightarrow$ $(Tx_n)_n$ has no convergent subsequence, although $(x_n)_n$ is bounded

contradiction to T compact

$\Rightarrow$ codim ran $A < \infty$ $\square$

11.8 Theorem: Let X be a Banach space and $T \in \mathcal{K}(X)$. Put $A := I - T$. Then we have:

(i) $(\ker A^n)_{n \geq 1}$ is a monotone increasing sequence of finite-dimensional subspaces of X and there is an $n \geq 1$ s. th.

$$\ker A^n = \ker A^{n+1} =: N$$

(ii) $(\operatorname{ran} A^m)_{m \geq 1}$ is a monotone decreasing sequence of closed subspaces with finite codimension and we have (with n from (i)):

$$\operatorname{ran} A^n = \operatorname{ran} A^{n+1} =: R$$

(iii) We have: $X = N \oplus R$ and

$A : R \to R$ is isomorphism

11.9. Example: In finite dimensions this corresponds to Jordan form

$$T = \begin{pmatrix} 1 & 1 & 0 & & \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 1 & & \\ & 0 & & \lambda & 1 \\ & & & 0 & \lambda \end{pmatrix}$$

$\lambda_1 = 1$

$\lambda_2 = \lambda \neq 1$

$$\Rightarrow A = I - T = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \\ & 0 & & \tilde{\lambda} & 1 \\ & & & 0 & \tilde{\lambda} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} N \\ \left.\vphantom{\begin{matrix}0\\0\end{matrix}}\right\} R \end{matrix}$$

$\tilde{\lambda} = 1 - \lambda \neq 0$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ & & & \tilde{\lambda}^2 & * \\ & 0 & & 0 & \tilde{\lambda}^2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ & & & \tilde{\lambda}^3 & * \\ & 0 & & 0 & \tilde{\lambda}^3 \end{pmatrix}$$

$\Rightarrow$ $\ker A = \mathbb{C} \oplus 0 \oplus 0 \oplus 0 \oplus 0$

$\ker A^2 = \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus 0 \oplus 0$

$\ker A^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus 0 = \ker A^4 =: N$

$\operatorname{ran} A = \mathbb{C} \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}$

$\operatorname{ran} A^2 = \mathbb{C} \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}$

$\operatorname{ran} A^3 = 0 \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C} = \operatorname{ran} A^4 =: R$

$\mathbb{C}^5 = N \oplus R$

$A : R \to R$ Isomorphism

Proof: note: $A^n = (I-T)^n = I - T_n$

where T_n compact

(by induction: $A^n = (I-T)\underbrace{(I-T)^{n-1}}_{I - T_{n-1}}$

$= I - \underbrace{T_{n-1} - T + T \cdot T_{n-1}}_{=: T_n \text{ compact}}$)

thus we have statements from 10.2. also for A^n, i.e.

- $\ker A^n$ finite-dimensional
- $\operatorname{ran} A^n$ closed and of finite codimension

monotonicity of sequences:

$\ker A^n \subset \ker A^{n+1}$: $A^n x = 0 \Rightarrow A^{n+1} x = A(A^n x) = 0$

$\operatorname{ran} A^n \supset \operatorname{ran} A^{n+1}$: $y = A^{n+1} x = A^n (Ax)$

to show: $\exists n$ s.th. $\ker A^n = \ker A^{n+1}$

Assume $\ker A^n \neq \ker A^{n+1}$ $\forall n$

$\Rightarrow \exists$ sequence $(x_n)_n$ s.th. $x_n \in \ker A^n$, but $x_n \notin \ker A^{n-1}$

By 8.1, we can choose x_n in such a way that

$\|x_n\| = 1$ and $\operatorname{dist}(x_n, \ker A^{n-1}) \geq \frac{1}{2}$

$(x_n)_n$ bounded $\xRightarrow{T \text{ compact}}$ (Tx_n) has convergent subsequence

but: $\|Tx_n - Tx_m\| = \| x_n \underbrace{- Ax_n - x_m + Ax_m}_{\in \ker A^{n-1} \quad (\text{if } n > m)} \|$

$\geq \operatorname{dist}(x_n, \ker A^{n-1})$

$\geq \frac{1}{2}$ contradiction

$\Rightarrow \exists n : \ker A^n = \ker A^{n+1} = \ker A^{n+2} = \dots$

$(A^{n+2} x = 0 \Rightarrow A^{n+1}(Ax) = 0 \Rightarrow Ax \in \ker A^{n+1} = \ker A^n$

$\Rightarrow A^n A x = 0$

in the same way for $\operatorname{ran} A^m$: Assume $\operatorname{ran} A^m \neq \operatorname{ran} A^{m+1}$ $\forall$

$\Rightarrow \exists$ sequence $(x_m)_m$ s.th. $x_m \in \operatorname{ran} A^m$, $x_m \notin \operatorname{ran} A^{m+1}$

$\|x_m\| = 1$, $\operatorname{dist}(x_m, A^{m+1}) \geq \frac{1}{2}$

$(x_m)_m$ bounded $\overset{T \text{ compact}}{\Longrightarrow}$ $\exists$ convergent subsequence of (Tx_m)

but: $\| Tx_n - Tx_m \| = \| \underbrace{x_n - Ax_n + Ax_m}_{\in \operatorname{ran} A^{m+1}} - x_m \|$ (if $n > m$)

$\geq \operatorname{dist}(x_m, A^{m+1})$

$\geq \frac{1}{2}$ contradiction

$\Rightarrow \exists m : \operatorname{ran} A^m = \operatorname{ran} A^{m+1} = \operatorname{ran} A^{m+2} = \dots$

Let n_0, m_0 be the smallest n, m with above properties.

We will show: $n_0 \geq m_0$

first observe: $\ker A^{n_0} \cap \operatorname{ran} A^{n_0} = \{0\}$

(because: $y \in \ker A^{n_0} \cap \operatorname{ran} A^{n_0}$

$\Rightarrow y = A^{n_0} x \Rightarrow 0 = A^{n_0} y = A^{2n_0} x$

$\Rightarrow x \in \ker A^{2n_0} = \ker A^{n_0}$

$\Rightarrow y = A^{n_0} x = 0$)

Assume $n_0 < m_0$, i.e.

$\operatorname{ran} A^{m_0} \underset{\neq}{\subset} \operatorname{ran} A^{m_0 - 1} \subset \operatorname{ran} A^{n_0}$ ($n_0 \leq m_0 - 1$)

$\Rightarrow \exists z \notin \operatorname{ran} A^{m_0}$ s.th. $z \in \operatorname{ran} A^{m_0-1} \subset \operatorname{ran} A^{n_0}$

$\Downarrow$

$Az \in \operatorname{ran} A^{m_0} = \operatorname{ran} A^{m_0+1}$

$\Rightarrow Az = Ax$ where $x \in \operatorname{ran} A^{m_0} \subset \operatorname{ran} A$

$\Rightarrow A(z - x) = 0$

i.e. $z - x \in \ker A \subset \ker A^{n_0}$

but: $z, x \in \operatorname{ran} A^{n_0}$

$\Rightarrow z - x \in \operatorname{ran} A^{n_0} \cap \ker A^{n_0} = \{0\}$

$\Rightarrow z = x \in \operatorname{ran} A^{m_0}$ contradiction

$\Rightarrow n_0 \geq m_0$

(iii) $N := \ker A^{n_0}$

$R := \operatorname{ran} A^{m_0} = \operatorname{ran} A^{n_0}$

we have shown above: $N \cap R = \{0\}$

to show: $N + R = X$

Consider $x \in X$

(want to write it as $x = \tilde{y} + y$, $\tilde{y} \in \ker A^{n_0}$, $y \in \operatorname{ran} A^{n_0}$

$\Rightarrow A^{n_0} x = A^{n_0} y$)

$A^{n_0} x \in \operatorname{ran} A^{n_0} = \operatorname{ran} A^{2n_0}$, i.e. $A^{n_0} x = A^{n_0} y$

where $y \in \operatorname{ran} A^{n_0} = R$

$\Rightarrow A^{n_0}(x - y) = 0$

$\Rightarrow x - y \in \ker A^{n_0} = N$

$$\Rightarrow x = \underbrace{(x-y)}_{\substack{\cap \\ N}} + \underset{\substack{\cap \\ R}}{y}$$

$\Rightarrow X = N \oplus R$

remains to show: $A: R \to R$ isomorphism

note: $A|_R$ is surjective, since
$$A(R) = A(\operatorname{ran} A^{n_0}) = \operatorname{ran} A^{n_0+1} = \operatorname{ran} A^{n_0} = R$$

and injective, since
$$\ker A|_R \subset \underbrace{\ker A}_{\subset \ker A^{n_0} = N} \cap R \subset N \cap R = \{0\}$$

thus: $A: R \to R$ bijective

$$\left.\begin{array}{l} A \text{ bounded} \\ R \text{ Banach space} \\ (\text{since closed}) \end{array}\right\} \overset{7.5.}{\Longrightarrow} A: R \to R \text{ isomorphism}$$

$\square$