Proof of Theorem 11.5 (ii) Consider $\lambda \in \sigma(T) \setminus \{0\}$

$$\Rightarrow (T - \lambda I) = \underbrace{-\lambda\,(I - \underbrace{\tfrac{1}{\lambda}T}_{\text{compact}})}_{=:A_\lambda}$$

$\overset{11.8}{\Rightarrow}$ $X = N_\lambda \oplus R_\lambda$ s.th.

$A_\lambda : R_\lambda \to R_\lambda$ isomorphism

and

$N_\lambda = \ker A_\lambda^{n_0}$,

$\ker A_\lambda^{n_0 - 1} \underset{\uparrow \text{ if } n_0 \geq 1}{\subsetneq} \ker A_\lambda^{n_0} = \ker A_\lambda^{n_0+1}$

$\lambda \in \sigma(T) \Rightarrow N_\lambda \neq \{0\}$ (otherwise: $R_\lambda = X$ and (thus $n_0 \geq 1$) $A_\lambda : R_\lambda \to R_\lambda$ isom.)

$\Rightarrow \exists\, x \in \ker A_\lambda^{n_0} \setminus \ker A_\lambda^{n_0 - 1}$

i.e. $y := A_\lambda^{n_0 - 1} x \neq 0$ and $A_\lambda y = A_\lambda^{n_0} x = 0$

$\Rightarrow \ker A_\lambda \neq \{0\}$

$\quad\ \ \|$

$\ker (T - \lambda I)$ $\qquad \Rightarrow \lambda \in \sigma_p(T)$

furthermore: $\ker(T - \lambda I) = \ker A_\lambda \subseteq \ker A_\lambda^{n_0} = N_\lambda$

and $\dim N_\lambda < \infty$ (by 11.8)

$\Rightarrow \dim \ker (T - \lambda I) < \infty$

(iii) Consider $\lambda \in \mathbb{C} \setminus \{0\}$

$T - \lambda I$ injective $\Rightarrow \lambda \notin \sigma_p(T)$

$\overset{(ii)}{\Rightarrow} \lambda \notin \sigma(T)$

$\Rightarrow T - \lambda I$ bijective, in particular surjective

$T - \lambda I$ surjective $\Rightarrow A_\lambda = I - \frac{1}{\lambda} T$ surjective

$\Rightarrow A_\lambda^m$ surjective $\forall m = 1, 2, \dots$

$\Rightarrow R = \operatorname{ran} A_\lambda^{m_0} = X$

$$\Big\Downarrow \; A_\lambda : \underset{\overset{\shortparallel}{X}}{R} \to \underset{\overset{\shortparallel}{X}}{R} \quad \text{isom}$$

A_λ injective

$\Rightarrow T - \lambda I$ injective

(i) We will show: Each $\lambda \in \sigma(T) \setminus \{0\}$ is isolated in $\sigma(T)$

i.e. $\exists \varepsilon > 0 : \mathcal{U}_\varepsilon(\lambda) \cap \sigma(T) = \{\lambda\}$

i.e. we have to show:

$T - (\lambda + \mu) I$ is invertible for $0 < |\mu| \leq \varepsilon$

We have: $T - (\lambda + \mu) I = \underbrace{(T - \lambda I)}_{\text{we know this}} - \mu I$

$X = N_\lambda \oplus R_\lambda$ and

$T - \lambda I : R_\lambda \to R_\lambda$ isomorphism

and

$\exists n : (T - \lambda I)^n \equiv 0$ on N_λ

note: $(T - \lambda I) - \mu I : R_\lambda \to R_\lambda$

$N_\lambda \to N_\lambda$

hence, it suffices to show that

$(T - (\lambda + \mu) I)|R_\lambda$ and $(T - (\lambda + \mu) I)|N_\lambda$

are both invertible.

This follows from

11.10 Lemma: Let X be a Banach space and $A \in B(X)$.

1) If A is nilpotent (i.e. $\exists n : A^n = 0$), then

$\sigma(A) = \{0\}$

(i.e. $A - \mu I$ is invertible $\forall \mu \neq 0$)

2) If $\lambda \notin \sigma(A)$, then $\exists \varepsilon > 0$ s.th.

$\lambda + \mu \notin \sigma(A) \quad \forall |\mu| \leq \varepsilon$

(i.e. $\varrho(A)$ is open, i.e. $\sigma(A)$ is closed)

Proof: note: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ if this makes sense

1) $(A-\mu I)^{-1} = -\mu^{-1}(I-\mu^{-1}A)^{-1}$ $(\mu \neq 0)$

$$= -\mu^{-1} \sum_{k=0}^{\infty} \frac{A^k}{\mu^k}$$

$$= -\mu^{-1} \sum_{k=0}^{n-1} \frac{A^k}{\mu^k} \quad \text{is inverse}$$

2) $\lambda \notin \sigma(A) \Rightarrow (A-\lambda I)^{-1}$ exists

$$A-(\lambda+\mu)I = (A-\lambda I)\,[I-\mu(A-\lambda I)^{-1}]$$

$$\Rightarrow (A-(\lambda+\mu)I)^{-1} = [\;]^{-1}(A-\lambda I)^{-1}$$

if $[\;]^{-1}$ exists

Put $B := \mu(A-\lambda I)^{-1}$

$|\mu|$ suff. small $\Rightarrow \|B\| < 1$

$$\Rightarrow (I-B)^{-1} = \underbrace{\sum_{n=0}^{\infty} B^n}_{\text{converges, since } \|B\|<1}$$

thus: $|\mu|$ suff. small $\Rightarrow (A-(\lambda+\mu)I)^{-1} \in B(X)$ exist

$\Rightarrow \lambda+\mu \notin \sigma(A)$

end of proof of ~~10.17~~ 11.5

(i) 11.10 $\Rightarrow$ all $\lambda \in \sigma(T) \setminus \{0\}$ are isolated in $\sigma(T)$

$\Rightarrow$ only possible limit point is 0

$\Rightarrow$ $\sigma(T)$ is countable (finite or countably infinite)

Assume $0 \notin \sigma(T)$, i.e. T bijective

$\xRightarrow[\text{Mapp. Th.}]{\text{Inverse}}$ $T^{-1} \in B(X)$

$\Rightarrow$ $I = T\,T^{-1} \in \mathcal{K}(X)$, since $\mathcal{K}(X)$ ideal

$\Rightarrow$ X finite-dimensional

thus, for $\dim X = \infty$: $\sigma(T)$ compact, since

closed $(0 \in \sigma(T))$

bounded $(\sigma(T) \subset \{\lambda \mid |\lambda| \le \|T\|$

$\uparrow$

$\lambda \in \sigma_p(T) \Rightarrow |\lambda| \le \|T\|$

for $\dim X < \infty$: $\sigma(T)$ compact, too, since finite set

$\square$