15. Spectral Theorem for Normal Operators on Hilbert Spaces

15.1 Motivation: 1) Consider

finite-dimensional case,

$\dim \mathcal{H} < \infty$ and $A = A^* \in B(\mathcal{H})$

Then the spectrum only consists of eigenvalues

$\sigma(A) = \{\lambda_1 < \lambda_2 < .. < \lambda_k\} \subset \mathbb{R}$

Denote

$\mathcal{H}_\lambda := \ker(A - \lambda 1)$ eigenspace of $\lambda \in \sigma(A)$

$P_\lambda :=$ orthogonal projection onto $\mathcal{H}_\lambda$

Then we can formulate spectral theorem in the form:

- $\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathcal{H}_\lambda$ i.e., $\sum_{\lambda \in \sigma(A)} P_\lambda = 1$
  $P_\lambda \cdot P_\mu = 0 \ (\lambda \neq \mu)$
- $A = \sum_{\lambda \in \sigma(A)} \lambda \cdot P_\lambda$

i.e.

$$A = \begin{array}{cccc|l} \boxed{\begin{matrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_1\end{matrix}} & & & & \} \mathcal{H}_{\lambda_1} \\ & \boxed{\begin{matrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_2\end{matrix}} & & & \} \mathcal{H}_{\lambda_2} \\ & & \ddots & & \vdots \\ & & & \boxed{\begin{matrix}\lambda_k & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{matrix}} & \} \mathcal{H}_{\lambda_k} \\ \hline \underbrace{\qquad}_{\mathcal{H}_{\lambda_1}} & \underbrace{\qquad}_{\mathcal{H}_{\lambda_2}} & \cdots & \underbrace{\qquad}_{\mathcal{H}_{\lambda_3}} & \end{array}$$

2) There is analoguous representation also for <u>compact</u> s.a. operators for $\dim \mathcal{H} = \infty$. Note that for compact operator T each $\lambda \in \sigma(T)$, $\lambda \neq 0$ is an eigenvalue

3) Our goal: generalize this to arbitrary bounded operators in case $\dim \mathcal{H} = \infty$

Problem: in general, we do not have eigenvectors for $\lambda \in \sigma(A)$, then $P_\lambda = 0$

15.2. Example: multiplication operator

$$\mathcal{H} = L^2[0,1]$$
$$= \{ f : [0,1] \to \mathbb{C} \mid \int_0^1 |f(t)|^2 dt < \infty \}$$

$$A : \mathcal{H} \to \mathcal{H}$$
$$f \mapsto Af \quad \text{with } (Af)(t) = t f(t)$$

then: $$\langle g, Af \rangle = \int_0^1 g(t) \overline{t f(t)} \, dt$$
$$= \int_0^1 t g(t) \overline{f(t)} \, dt$$
$$= \langle Ag, f \rangle \quad \forall f, g \in \mathcal{H}$$

$$\Rightarrow A = A^*$$

$$\|Af\|^2 = \int_0^1 |t f(t)|^2 dt$$
$$\leq \int_0^1 |f(t)|^2 dt = \|f\|^2$$

$$\Rightarrow \|A\| \leq 1 \quad \Rightarrow A \in B(\mathcal{H})$$

easy to see: $\sigma(A) = [0,1]$,

but there exist no eigenvectors

$$A f = \lambda f \qquad (0 \leq \lambda \leq 1)$$

$$\Rightarrow t f(t) = \lambda f(t) \qquad \text{a.s.}$$

$$\Rightarrow f = 0 \qquad (f \mathrel{\hat{=}} \text{delta-fct at } \lambda \notin \mathcal{H})$$

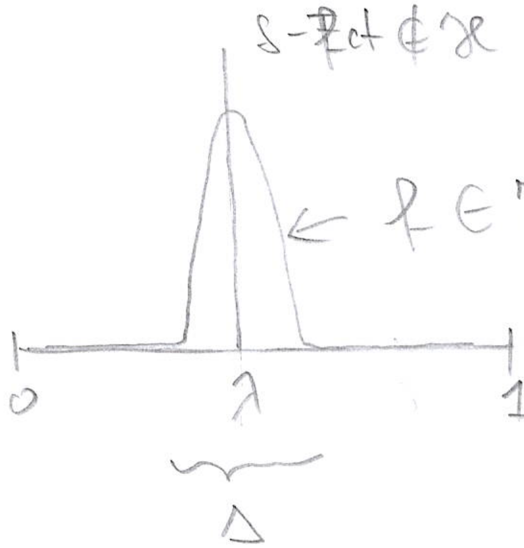
Hence the representation

$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$ does not hold,

since all $P_\lambda = 0$

however: we have kind of replacement for eigenvectors, namely

approximate eigenvectors $\mathrel{\hat{=}} \mathcal{H}_\Delta$ for intervals Δ

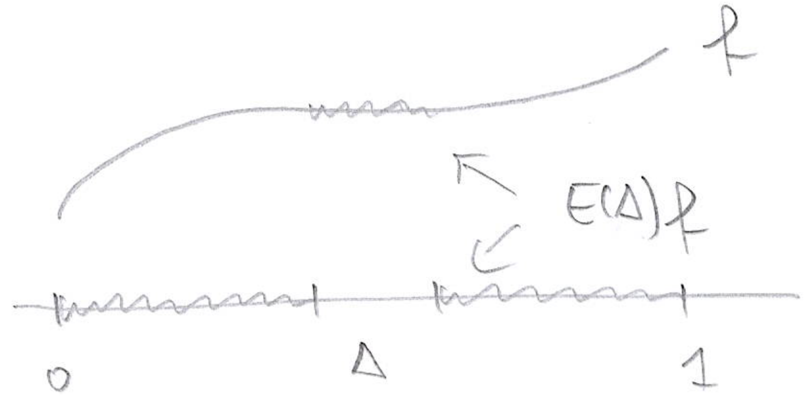


$\leftarrow f \in \mathcal{H}_\Delta \mathrel{\hat{=}}$ fcts which are localized in Δ

$$\left.\begin{array}{l} f \in \mathcal{H}_\Delta \\ |\Delta| \text{ small} \end{array}\right\} \Rightarrow A f \approx \lambda f$$

Let $E(\Delta) : \mathcal{H} \to \mathcal{H}_\Delta$ (for $\Delta \subset \mathbb{R}$ interval)

be orthogonal projection onto $\mathcal{H}_\Delta$



those $E(\Delta)$ allow continuous decomposition of our Hilbert space:

- $\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow E(\Delta_1) \cdot E(\Delta_2) = 0$
- $\Delta = \Delta_1 \dot{\cup} \Delta_2 \Rightarrow E(\Delta) = E(\Delta_1) + E(\Delta_2)$

$\Delta_1 \cap \Delta_2 = \emptyset$

and thus

$$\mathbb{R} = \dot{\bigcup_{i=1}^{n}} \Delta_i \Rightarrow 1 = \sum_{i=1}^{n} E(\Delta_i)$$

such decompositions can be refined, but in this case there is no finest one.

Do we have an analogous representation for the operator A as in finite-dim. case?

$$\left.\begin{array}{l}|\Delta| \text{ small} \\ f \in \mathcal{H}_\Delta\end{array}\right\} \Rightarrow A f \approx \lambda f \quad (\text{where } \lambda \in \Delta)$$

thus: $$A f = \sum_i \underbrace{A \underbrace{E(\Delta_i) f}_{\in \mathcal{H}_{\Delta_i}}}_{\approx \lambda_i E(\Delta_i) f} \qquad (\lambda_i \in \Delta_i)$$

$$\Rightarrow A \mathrel{\hat{=}} \lim_{|\Delta_i| \to 0} \sum_i \lambda_i E(\Delta_i)$$

$$\mathrel{\hat{=}} \int \lambda \, dE(\lambda)$$

with $E(\lambda) = E(-\infty, \lambda]$

This is an "operator-valued" integral with respect to the "spectral measure"

$\Delta \mapsto E(\Delta)$

In order to make this rigorous and prove a spectral theorem in this form we need two ingredients:

i) for a spectral measure $E(\cdot)$ we need a general theory to define and handle $\int \lambda \, dE(\lambda)$, and more general $\int f(\lambda) dE(\lambda)$

ii) for a given $A = A^* \in B(\mathcal{H})$
(and more general for normal A)
we have to show the existence
of spectral measure $E(\cdot)$ s.th.

$$A = \int \lambda \, dE(\lambda) \quad \text{and}$$

$$f(A) = \int f(\lambda) \, dE(\lambda)$$

This shows that finding spectral measure for A is related to functional calculus for A, since
for $f = 1_\Delta$ $\left[\text{i.e., } f(t) = \begin{cases} 1 & t \in \Delta \\ 0 & t \notin \Delta \end{cases}\right]$
we must have

$$1_\Delta(A) = \int 1_\Delta(\lambda) \, dE(\lambda) = E(\Delta),$$

thus the spectral measure of A should be given by characteristic functions of A, which we can deal with via the measurable functional calculus ∵

<u>15.3. Def.:</u> Let $A \in B(\mathcal{H})$ be normal and $\tilde{\Phi}$ the measurable functional calculus of A, as in 14.10. Let $\mathfrak{B}(\sigma(A))$ be the Borel-measurable subsets of $\sigma(A)$ and, for $M \in \mathfrak{B}(\sigma(A))$,

$$1_M(\lambda) := \begin{cases} 1 & \lambda \in M \\ 0 & \lambda \notin M \end{cases}$$

We define then

$$E : \mathfrak{B}(\sigma(A)) \to B(\mathcal{H})$$

by

$$E(M) := \tilde{\Phi}(1_M) = 1_M(A).$$

<u>15.4. Proposition:</u> The mapping

$$E : \mathfrak{B}(K) \to B(\mathcal{H})$$

from 15.3., where $K := \sigma(A)$, has the following properties:

a) Each $E(M)$, for $M \in \mathfrak{B}(K)$, is an orthogonal projection

$$E(\emptyset) = 0$$

$$E(K) = 1$$

b) $E(M_1 \cap M_2) = E(M_1)E(M_2)$

$\forall M_1, M_2 \in \mathfrak{a}(k)$

c) $E(M_1 \cup M_2) = E(M_1) + E(M_2)$

$\forall M_1, M_2 \in \mathfrak{a}(k)$ s.th. $M_1 \cap M_2 = \phi$

d) For each $x \in \mathcal{H}$,

$E_x : M \mapsto E_x(M) := \langle E(M)x, x \rangle$

is a measure on k

Proof: a) – c) follow directly by functional calculus and corresponding properties for characteristic functions

e.g.: $E(M)^2 = \tilde{\Phi}(1_M) \cdot \tilde{\Phi}(1_M)$

$$= \tilde{\Phi}(\underbrace{1_M \cdot 1_M}_{= 1_M})$$

$$= E(M)$$

$E(M)^* = \tilde{\Phi}(1_M)^* = \tilde{\Phi}(\overline{1}_M) = \tilde{\Phi}(1_M)$

$= E(M)$

d) follows by general measure / integration theory and the Riesz representation theorem, which says that the positive bounded linear functionals on $C(K)$ correspond to measures on K, via integration.

The measure E_x integrates then functions via

$$\int f(t)\, dE_x(t) = \langle f(A)x, x \rangle$$

and (apart from all the technical details) one has to check the positivity of this:

$$f \geq 0 \text{ on } K \overset{!}{\Longrightarrow} \int f(t)\, dE_x(t) \geq 0$$

$$\Downarrow$$

$g := \sqrt{f}$ exists and $f = g^2$

$$\Rightarrow \int f(t)\, dE_x(t) = \langle f(A)x, x \rangle$$

$$= \langle g^2(A)x, x \rangle = \langle g(A)g(A)x, x \rangle$$

$$= \langle g(A)x, \underbrace{g(A)^* x}_{g(A)} \rangle \geq 0$$

$\square$

<u>15.5. Def.</u>: A map, for $K \subset \mathbb{C}$ compact,

$$E : \sigma(K) \to B(\mathcal{H})$$

that satisfies the conditions a) - d) from Prop. 15.4, is called a <u>spectral measure</u> on K.

Other names are: <u>projection-valued measure</u> or <u>resolution of the identity</u>

<u>15.6. Theorem</u>: Let E be a spectral measure on K. Then there exists for each $h \in M_\infty(K)$ a uniquely determined operator

$$\int h(\lambda)\, dE(\lambda) \in B(\mathcal{H}) \qquad \text{s.th.}$$

(i) $\langle \int h(\lambda)\, dE(\lambda)\, x, x \rangle = \int h(\lambda)\, dE_x(\lambda) \quad \forall x \in \mathcal{H}$

ii) $\| \int h(\lambda)\, dE(\lambda)\, x \|^2 = \int |h(\lambda)|^2\, dE_x(\lambda) \quad \forall x \in \mathcal{H}$

The mapping

$$h \mapsto \int h(\lambda)\, dE(\lambda)$$

is a $*$-homomorphism from $\mathcal{M}_\infty(K)$ into $B(\mathcal{H})$.

<u>15.7 Remark:</u> The mapping

$$h \mapsto \int h(\lambda)\, dE(\lambda)$$

is of course the functional calculus for A, if $E(\lambda)$ comes from A. The point here is that we can define operators with respect to any spectral measure as integrals and use this then to write normal operators in such a form.

<u>15.8. Spectral Theorem for Normal Operators:</u>

Let $A \in B(\mathcal{H})$ be normal. Then there exists a uniquely determined spectral measure E on $\sigma(A)$ s.th.

$$A = \int \lambda\, dE(\lambda).$$

Furthermore one has:

$$h(A) = \int h(\lambda)\, dE(\lambda) \quad \forall h \in \mathcal{M}_\infty(\sigma(A))$$

<u>15.9. Example</u>: Consider multiplication operator from 15.2

$\mathcal{H} = L^2[0,1]$

$(Af)(t) = t\,f(t)$

Define $E: \sigma[0,1] \to B(\mathcal{H})$ by

$$E(M)f = 1_M \cdot f \quad , \text{ i.e.}$$

$$(E(M)f)(t) = \begin{cases} f(t) & t \in M \\ 0 & t \notin M \end{cases}$$

$\Rightarrow$ E spectral measure

in particular

$$E_f(M) = \langle E(M)f, f \rangle$$
$$= \int_0^1 1_M(t) f(t) \overline{f(t)}\, dt$$
$$= \int_M |f(t)|^2\, dt$$

$\Rightarrow \quad dE_f(t) = |f(t)|^2\, dt$

and thus

$$\langle \int \lambda \, dE(\lambda) f, f \rangle = \int \lambda \, dE_f(\lambda)$$

$$= \int \lambda \, |f(\lambda)|^2 \, d\lambda$$

$$= \int (Af)(\lambda) \, \overline{f(\lambda)} \, d\lambda$$

$$= \langle Af, f \rangle \quad \forall f \in \mathcal{H}$$

$$\Rightarrow A = \int \lambda \, dE(\lambda)$$

15.10. Proposition: A normal operator $A \in B(\mathcal{H})$ is compact if and only if the following two conditions are satisfied:

a) $\sigma(A) \setminus \{0\}$ has no cluster point

b) for each $\lambda \neq 0$ we have

$$\dim \ker (A - \lambda \cdot 1) < \infty$$

Proof: "$\Rightarrow$": shown in 11.5.

"$\Leftarrow$": a) $\Rightarrow$ $\sigma(A)$ is countable; write $\sigma(A) \setminus \{0\} = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ with

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$$

Define f_n by

$$f_n(\lambda) = \begin{cases} \lambda_i & \text{if } \lambda = \lambda_i \quad i \leq n \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda \in \sigma(A)$

note: $f_n \in C(\sigma(A))$

and

$|\lambda - f_n(\lambda)| \leq |\lambda_n| \quad \forall \lambda \in \sigma(A)$

thus $f_n \xrightarrow{n \to \infty}$ identity fct.

$z : \lambda \mapsto \lambda$

in $C(\sigma(A))$ w.r.t. $\|\cdot\|_{sup}$

Let E be spectral measure of A, i.e.

$A = \int \lambda \, dE(\lambda)$; then

$$f_n(A) = \int f_n(\lambda) \, dE(\lambda)$$

$$= \sum_{i=1}^{n} \lambda_i E(\{\lambda_i\}) \quad \text{compact,}$$

since $E(\{\lambda_i\})$ = orth. proj. onto $\ker(A - \lambda_i 1)$

has finite rank by b)

furthermore:

$$\| A - f_n(A) \| = \| z - f_n \|_{C(\sigma(A))} \xrightarrow{n \to \infty} 0$$

thus:

$$\left.\begin{array}{l} f_n(A) \text{ compact } \forall n \\ f_n(A) \to A \text{ in } \|\cdot\| \end{array}\right\} \overset{10.5}{\Longrightarrow} \quad A \text{ compact}$$

$\square$