15. Spectral Theorem for Normal Operators on Hilbert Spaces

15.1 Motivation: 1) Consider finite-dimensional case, $\dim \mathcal{H} < \infty$ and $A = A^* \in B(\mathcal{H})$

Then the spectrum only consists of eigenvalues
$\sigma(A) = \{\lambda_1 < \lambda_2 < \ldots < \lambda_k\} \subset \mathbb{R}$

Denote
$\mathcal{H}_\lambda := \ker(A - \lambda 1)$ eigenspace of $\lambda \in \sigma(A)$

$P_\lambda :=$ orthogonal projection onto $\mathcal{H}_\lambda$

Then we can formulate spectral theorem in the form:

- $\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathcal{H}_\lambda$ i.e., $\sum_{\lambda \in \sigma(A)} P_\lambda = 1$

  $P_\lambda \cdot P_\mu = 0 \ (\lambda \neq \mu)$
- $A = \sum_{\lambda \in \sigma(A)} \lambda \cdot P_\lambda$

i.e.

$$A = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_1\end{matrix}} & & & \\ & \boxed{\begin{matrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}\lambda_k & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{matrix}} \end{pmatrix} \\ \quad \underbrace{\qquad\qquad}_{\mathcal{H}_{\lambda_1}} \; \underbrace{\qquad\qquad}_{\mathcal{H}_{\lambda_2}} \cdots \underbrace{\qquad\qquad}_{\mathcal{H}_{\lambda_3}} \end{array} \begin{array}{l} \}\ \mathcal{H}_{\lambda_1} \\ \}\ \mathcal{H}_{\lambda_2} \\ \ \vdots \\ \}\ \mathcal{H}_{\lambda_k} \end{array}$$

2) There is analoguous representation also for <u>compact</u> s.a. operators for $\dim \mathcal{H} = \infty$. Note that for compact operator T each $\lambda \in \sigma(T)$, $\lambda \neq 0$ is an eigenvalue

3) Our goal: generalize this to arbitrary bounded operators in case $\dim \mathcal{H} = \infty$

Problem: in general, we do not have eigenvectors for $\lambda \in \sigma(A)$, then $P_\lambda = 0$

15.2. Example: multiplication operator

$\mathcal{H} = L^2[0,1]$

$= \{ f : [0,1] \to \mathbb{C} \mid \int_0^1 |f(t)|^2 dt < \infty \}$

$A : \mathcal{H} \to \mathcal{H}$

$f \mapsto Af$ with $(Af)(t) = t f(t)$

then: $\langle g, Af \rangle = \int_0^1 g(t) \overline{t f(t)} \, dt$

$= \int_0^1 t g(t) \overline{f(t)} \, dt$

$= \langle Ag, f \rangle \quad \forall f, g \in \mathcal{H}$

$\Rightarrow A = A^*$

$\|Af\|^2 = \int_0^1 |t f(t)|^2 dt$

$\leq \int_0^1 |f(t)|^2 dt = \|f\|^2$

$\Rightarrow \|A\| \leq 1 \quad \Rightarrow A \in B(\mathcal{H})$

easy to see: $\sigma(A) = [0,1]$,

but there exist no eigenvectors

$A f = \lambda f \qquad (0 \le \lambda \le 1)$

$\Rightarrow t f(t) = \lambda f(t) \qquad$ a.s.

$\Rightarrow f = 0 \qquad$ ($f \mathrel{\hat{=}}$ delta-fct at λ $\notin \mathcal{H}$)

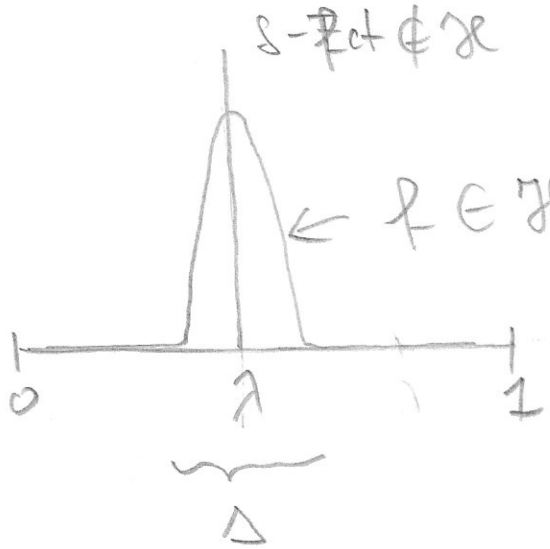
Hence the representation

$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$ does not hold,

since all $P_\lambda = 0$.

however: we have kind of replacement for eigenvectors, namely

approximate eigenvectors $\mathrel{\hat{=}} \mathcal{H}_\Delta$ for intervals Δ

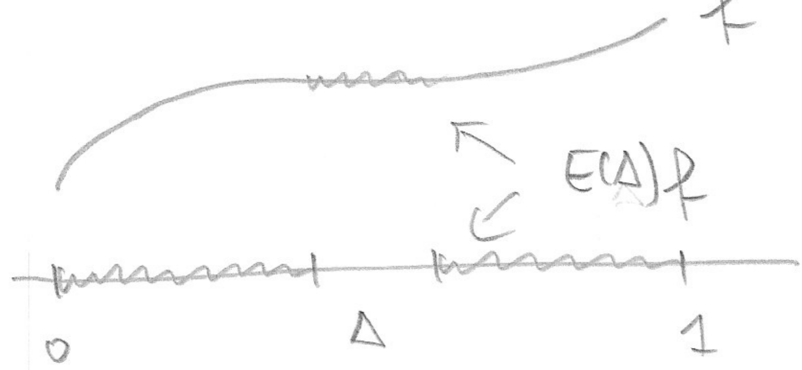


$\leftarrow f \in \mathcal{H}_\Delta \mathrel{\hat{=}}$ fcts which are localized in Δ

$$\left.\begin{array}{l} f \in \mathcal{H}_\Delta \\ |\Delta| \text{ small} \end{array}\right\} \Rightarrow A f \approx \lambda f$$

Let $E(\Delta): \mathcal{H} \to \mathcal{H}_\Delta$ (for $\Delta \subset \mathbb{R}$ interval)

be orthogonal projection onto $\mathcal{H}_\Delta$



those $E(\Delta)$ allow continuous decomposition of our Hilbert space:

- $\Delta_1 \cap \Delta_2 = \phi \implies E(\Delta_1) \cdot E(\Delta_2) = 0$
- $\Delta = \Delta_1 \dot\cup \Delta_2 \implies E(\Delta) = E(\Delta_1) + E(\Delta_2)$

  $\Delta_1 \cap \Delta_2 = \phi$

and thus

$$\mathbb{R} = \dot{\bigcup_{i=1}^{n}} \Delta_i \implies 1 = \sum_{i=1}^{n} E(\Delta_i)$$

such decompositions can be refined, but in this case there is no finest one.

Do we have an analogous representation for the operator A as in finite-dim. case?

$$\left.\begin{array}{l}|\Delta| \text{ small} \\ f \in \mathcal{H}_\Delta\end{array}\right\} \Rightarrow Af \approx \lambda f \quad (\text{where } \lambda \in \Delta)$$

thus: $$Af = \sum_i A \underbrace{\underbrace{E(\Delta_i) f}_{\in \mathcal{H}_{\Delta_i}}}_{\approx \lambda_i E(\Delta_i) f} \qquad (\lambda_i \in \Delta_i)$$

$$\Rightarrow A \mathrel{\hat{=}} \lim_{|\Delta_i| \to 0} \sum_i \lambda_i E(\Delta_i)$$

$$\mathrel{\hat{=}} \int \lambda \, dE(\lambda)$$

with $E(\lambda) = E(-\infty, \lambda]$

This is an "operator-valued" integral with respect to the "spectral measure"

$\Delta \mapsto E(\Delta)$

In order to make this rigorous and prove a spectral theorem in this form we need two ingredients:

i) for a spectral measure $E(\cdot)$ we need a general theory to define and handle $\int \lambda \, dE(\lambda)$, and more general $\int f(\lambda) \, dE(\lambda)$

ii) for a given $A = A^* \in B(\mathcal{H})$

(and more general for normal A)

we have to show the existence of spectral measure $E(\cdot)$ s.th.

$A = \int \lambda \, dE(\lambda)$ and

$f(A) = \int f(\lambda) \, dE(\lambda)$

This shows that finding spectral measure for A is related to functional calculus for A, since for $f = 1_\Delta$ $\left[\text{i.e., } f(t) = \begin{cases} 1 & t \in \Delta \\ 0 & t \notin \Delta \end{cases} \right]$

we must have

$1_\Delta(A) = \int 1_\Delta(\lambda) \, dE(\lambda) = E(\Delta)$,

thus the spectral measure of A should be given by characteristic functions of A, which we can deal with via the measurable functional calculus $\overset{\nabla}{\circ}$