<u>15.3. Def.:</u> Let $A \in B(\mathcal{H})$ be normal and $\tilde{\Phi}$ the measurable functional calculus of A, as in 14.10. Let $\sigma(\sigma(A))$ be the Borel-measurable subsets of $\sigma(A)$ and, for $M \in \sigma(\sigma(A))$,

$$1_M(\lambda) := \begin{cases} 1 & \lambda \in M \\ 0 & \lambda \notin M \end{cases}$$

We define then

$$E : \sigma(\sigma(A)) \to B(\mathcal{H})$$

by

$$E(M) := \tilde{\Phi}(1_M) = 1_M(A).$$

<u>15.4. Proposition:</u> The mapping

$$E : \sigma(K) \to B(\mathcal{H})$$

from 15.3., where $K := \sigma(A)$, has the following properties:

a) Each $E(M)$, for $M \in \sigma(K)$, is an orthogonal projection

$$E(\emptyset) = 0$$

$$E(K) = 1$$

b) $E(M_1 \cap M_2) = E(M_1)E(M_2)$

$\forall M_1, M_2 \in \mathfrak{a}(k)$

c) $E(M_1 \cup M_2) = E(M_1) + E(M_2)$

$\forall M_1, M_2 \in \mathfrak{a}(k)$ s.th. $M_1 \cap M_2 = \phi$

d) For each $x \in \mathcal{H}$,

$E_x : M \mapsto E_x(M) := \langle E(M)x, x \rangle$

is a measure on k

<u>Proof:</u> a) – c) follow directly by functional calculus and corresponding properties for characteristic functions

e.g.: $E(M)^2 = \tilde{\Phi}(1_M) \cdot \tilde{\Phi}(1_M)$

$$= \tilde{\Phi}(\underbrace{1_M \cdot 1_M}_{= 1_M})$$

$$= E(M)$$

$E(M)^* = \tilde{\Phi}(1_M)^* = \tilde{\Phi}(\overline{1_M}) = \tilde{\Phi}(1_M)$

$$= E(M)$$

(15-10)

d) follows by general measure / integration theory and the Riesz representation theorem, which says that the positive bounded linear functionals on $C(K)$ correspond to measures on K, via integration.

The measure E_x integrates then functions via

$$\int f(t)\, dE_x(t) = \langle f(A)x, x \rangle$$

and (apart from all the technical details) one has to check the positivity of this:

$$f \geq 0 \text{ on } K \overset{!}{\Longrightarrow} \int f(t)\, dE_x(t) \geq 0$$

$$\Downarrow$$

$g := \sqrt{f}$ exists and $f = g^2$

$$\Rightarrow \int f(t)\, dE_x(t) = \langle f(A)x, x \rangle$$

$$= \langle g^2(A)x, x \rangle = \langle g(A)g(A)x, x \rangle$$

$$= \langle g(A)x, \underbrace{g(A)^* x}_{g(A)} \rangle \geq 0$$

$\square$

15.5. Def.: A map, for $K \subset \mathbb{C}$ compact,

$$E : \sigma(K) \to B(\mathcal{H})$$

that satisfies the conditions a) – d) from Prop. 15.4, is called a spectral measure on K.

Other names are: projection-valued measure or resolution of the identity

15.6. Theorem: Let E be a spectral measure on K. Then there exists for each $h \in M_\infty(K)$ a uniquely determined operator

$$\int h(\lambda)\, dE(\lambda) \in B(\mathcal{H}) \quad \text{s.th.}$$

(i) $\langle \int h(\lambda)\, dE(\lambda)\, x, x \rangle = \int h(\lambda)\, dE_x(\lambda) \quad \forall x \in \mathcal{H}$

ii) $\| \int h(\lambda)\, dE(\lambda)\, x \|^2 = \int |h(\lambda)|^2\, dE_x(\lambda) \quad \forall x \in \mathcal{H}$

The mapping

$$h \mapsto \int h(\lambda)\, dE(\lambda)$$

is a $*$-homomorphism from $\mathcal{M}_\infty(\mathbb{K})$ into $B(\mathcal{H})$.

15.7 Remark: The mapping

$$h \mapsto \int h(\lambda)\, dE(\lambda)$$

is of course the functional calculus for A, if $E(\lambda)$ comes from A. The point here is that we can define operators with respect to any spectral measure as integrals and use this then to write normal operators in such a form.

15.8 Spectral Theorem for Normal Operators:
Let $A \in B(\mathcal{H})$ be normal. Then there exists a uniquely determined spectral measure E on $\sigma(A)$ s.th.

$$A = \int \lambda \, dE(\lambda).$$

Furthermore one has:

$$h(A) = \int h(\lambda)\, dE(\lambda) \quad \forall h \in \mathcal{M}_\infty(\sigma(A))$$

15.9. Example: Consider multiplication operator from 15.2

$\mathcal{H} = L^2[0,1]$

$(Af)(t) = t\, f(t)$

Define $E : \mathfrak{B}[0,1] \to B(\mathcal{H})$ by

$$E(M)f = 1_M \cdot f \quad , \text{ i.e.}$$

$$(E(M)f)(t) = \begin{cases} f(t) & t \in M \\ 0 & t \notin M \end{cases}$$

$\Rightarrow$ E spectral measure

in particular

$$\begin{aligned} E_f(M) &= \langle E(M)f, f\rangle \\ &= \int_0^1 1_M(t)\, f(t)\, \overline{f(t)}\, dt \\ &= \int_M |f(t)|^2\, dt \end{aligned}$$

$\Rightarrow$ $dE_f(t) = |f(t)|^2\, dt$

and thus

$$\langle \int \lambda \, dE(\lambda) f, f \rangle = \int \lambda \, dE_f(\lambda)$$
$$= \int \lambda \, |f(\lambda)|^2 \, d\lambda$$
$$= \int (Af)(\lambda) \, \overline{f(\lambda)} \, d\lambda$$
$$= \langle Af, f \rangle \quad \forall f \in \mathcal{H}$$

$$\Rightarrow A = \int \lambda \, dE(\lambda)$$

15.10. Proposition: A normal operator $A \in B(\mathcal{H})$ is compact if and only if the following two conditions are satisfied:

a) $\sigma(A) \setminus \{0\}$ has no cluster point

b) for each $\lambda \neq 0$ we have
$$\dim \ker (A - \lambda \cdot 1) < \infty$$

Proof: "$\Rightarrow$": shown in 11.5.

"$\Leftarrow$": a) $\Rightarrow$ $\sigma(A)$ is countable; write $\sigma(A) \setminus \{0\} = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ with
$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$$

Define f_n by

$$f_n(\lambda) = \begin{cases} \lambda_i & \text{if } \lambda = \lambda_i \quad i \leq n \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda \in \sigma(A)$

note: $f_n \in C(\sigma(A))$

and

$|\lambda - f_n(\lambda)| \leq |\lambda_n| \quad \forall \lambda \in \sigma(A)$

thus $f_n \xrightarrow{n \to \infty}$ identity fct.
$z : \lambda \mapsto \lambda$

in $C(\sigma(A))$ w.r.t. $\|\cdot\|_{sup}$

Let E be spectral measure of A, i.e.

$A = \int \lambda \, dE(\lambda)$; then

$$f_n(A) = \int f_n(\lambda) \, dE(\lambda)$$

$$= \sum_{i=1}^{n} \lambda_i \, E(\{\lambda_i\}) \quad \text{compact,}$$

since $E(\{\lambda_i\})$ = orth. proj. onto $\ker(A - \lambda_i 1)$
has finite rank by b)

furthermore:

$$\|A - f_n(A)\| = \|z - f_n\|_{C(\sigma(A))} \xrightarrow{n \to \infty} 0$$

thus:

$$\left.\begin{array}{l} f_n(A) \text{ compact } \forall n \\ f_n(A) \to A \text{ in } \|\cdot\| \end{array}\right\} \overset{10.5}{\Longrightarrow} A \text{ compact}$$

$\square$