14. The Stone - Weierstrass Theorem

14.1. Motivation: Classical Theorem of Weierstrass (1895) says that polynomials are dense in $C[0,1]$ w.r.t. $\|\cdot\|_\infty$ - norm.

There are many proofs which use concrete properties of polynomials, but Stone noticed (1948) that actually only very a few properties are essential and thus gave an important and far reaching generalization of the classical Weierstrass Theorem. (*)

14.2. Def: Let K be a compact metric (or topological) space. We consider the Banach space

$$C(K) := \{ f : K \to \mathbb{R} \mid f \text{ continuous} \}$$

equipped with sup-norm

$$\|f\|_\infty := \sup_{t \in K} |f(t)|$$

1) $A \subset C(K)$ is a <u>unital subalgebra</u> if

the constant fct $1 \in A$ and

$$\left.\begin{array}{l} f, g \in A \\ \alpha, \beta \in \mathbb{R} \end{array}\right\} \Rightarrow \begin{array}{l} \alpha f + \beta g \in A \\ f \cdot g \in A \end{array}$$

2) $A \subset C(K)$ <u>separates points</u> of K if

$\forall s,t \in K$ with $s \neq t$ $\exists f \in A$ s.th. $f(s) \neq f(t)$

<u>14.3. Remarks:</u> 1) $1 \in A$ and A subalgebra implies that all constant fcts are in A

2) Polynomials fulfill these conditions clearly. But that's all we have to know about them!

<u>14.4. Theorem of Stone-Weierstrass:</u> Let K be a compact metric (or topological) space and $A \subset C(K)$ a unital subalgebra which separates the points of K. Then A is dense in $C(K)$.

<u>14.5. Remark:</u> note that this is equivalent to:

$$\left.\begin{array}{l} A \text{ unital subalgebra} \\ A \text{ separates points} \\ A \text{ closed} \end{array}\right\} \Rightarrow A = C(K)$$

We will prove this formulation.

<u>14.6. Lemma:</u> Let $A \subset C(K)$ be a closed unital subalgebra. Then

i) $\left.\begin{array}{l} f \in A \\ f \geq 0 \end{array}\right\} \Rightarrow \sqrt{f} \in A$

ii) $f \in A \Rightarrow |f| \in A$

iii) $f, g \in A \Rightarrow \min(f,g) \in A$
$\max(f,g) \in A$

<u>Proof:</u> note

$|f| = \sqrt{f^2}$

and

$\min(f,g) = \frac{1}{2}(f+g-|f-g|)$

$\max(f,g) = \frac{1}{2}(f+g+|f-g|)$

thus (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

so we only have to show (i)

Consider without restriction $0 \leq f \leq 1$,

i.e. $f = 1-g$ with $0 \leq g \leq 1$

Taylor series

$$\Rightarrow \sqrt{f(t)} = \sqrt{1-g(t)} = 1 - \sum_{n=1}^{\infty} a_n g(t)^n \quad \forall t \in K$$

with

$$a_n = \frac{1}{2n-1} 2^{-2n+1} \binom{2n-1}{n} < c \cdot n^{-3/2} \quad \text{(Stirling)}$$

for some $c > 0$

$$\Rightarrow \left\| \sqrt{f} - \left(1 - \sum_{n=1}^{N} a_n g^n\right) \right\| = \left\| \sum_{n=N+1}^{\infty} a_n g^n \right\|$$

$$\leq \sum_{n=N+1}^{\infty} a_n \underbrace{\|g\|^n}_{\leq 1}$$

$$\leq c \cdot \sum_{n=N+1}^{\infty} n^{-3/2}$$

$$\xrightarrow{N \to \infty} 0$$

note that we have $\|f \cdot g\| \leq \|f\| \cdot \|g\|$ $\forall f, g \in C(K)$ i.e. $C(K)$ is "Banach algebra"

thus: $\underbrace{1 - \sum_{n=1}^{N} a_n g^n}_{\in A} \xrightarrow{N \to \infty} \sqrt{f}$ in $C(K)$

$\Rightarrow \sqrt{f} \in A$

since A closed

$\square$

<u>Proof of 14.4:</u> Let A be a closed unital subalgebra of $C(K)$ which separates points of K.

Given $f \in C(K)$, $\varepsilon > 0$

to show: $\exists\, g \in A$ with $\|f - g\| < \varepsilon$

Consider $s, t \in K$, $s \neq t$

A separates points $\Rightarrow \exists\, h \in A : h(s) \neq h(t)$

Put (for some $\lambda, \mu \in \mathbb{R}$):

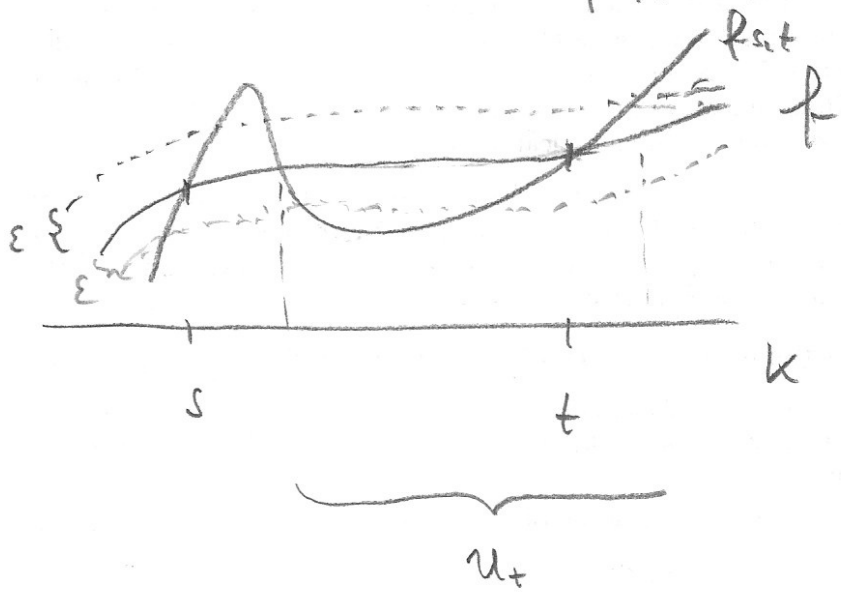
$$\tilde{h}(v) := \mu + (\lambda - \mu) \frac{h(v) - h(t)}{h(s) - h(t)} \qquad (v \in K)$$

$\Rightarrow \tilde{h} \in A$ and $\tilde{h}(s) = \lambda$, $\tilde{h}(t) = \mu$

thus: $s \neq t \Rightarrow \exists f_{s,t} \in A$ with

$$f_{s,t}(s) = f(s)$$

$$f_{s,t}(t) = f(t)$$



Fix s for the moment and put

$\mathcal{U}_t := \{ v \in K \mid f_{s,t}(v) < f(v) + \varepsilon \}$ open neighbourh of t

$\Rightarrow \bigcup_{t \in K} \mathcal{U}_t$ open cover of K

K compact $\Rightarrow \exists$ finite subcover,

i.e. $\exists\, t_1, \dots, t_n \in K$ with

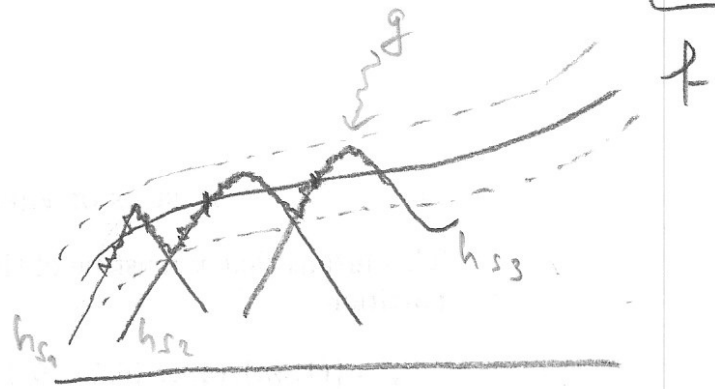
$$K = \bigcup_{i=1}^{n} \mathcal{U}_{t_i}$$

Put $h_s := \min_{1 \leq i \leq n} f_{s,t_i}$

$\Rightarrow h_s \in A$

$h_s(s) = f(s)$

$h_s < f + \varepsilon$



Put now

$V_s = \{ v \in K \mid h_s(v) > f(v) - \varepsilon \}$ open neighbourhood of s

$\Rightarrow \bigcup_{s \in K} V_s$ open cover of K

K compact $\Rightarrow$ $\exists$ finite subcover,

i.e. $\exists\ s_1, \dots, s_m \in K$ s.th. $K = \bigcup_{j=1}^{m} V_{s_j}$

Put $g := \max_{1 \le j \le m} h_{s_j}$

$\Rightarrow g \in A$

and: $f - \varepsilon < g < f + \varepsilon$, i.e. $\|f - g\| < \varepsilon$

$\square$

<u>14.7. Corollary</u>: Let X be a compact subset of $\mathbb{R}^n$. The algebra of all polynomials $p(x_1, \dots, x_n)$ in the coordinates is dense in $C_1(X)$.

($n = 1$ is classical Weierstrass Theorem)

<u>14.8. Corollary:</u> The set of all trigonometric polynomials, i.e. finite sums of the form

$$f(t) = a_0 + \sum_{k=1}^{N} (a_k \cos kt + b_k \sin kt)$$

is uniformly dense in the space of 2π-periodic continuous fcts on $\mathbb{R}$,

$$C_{per}[0,2\pi] = \{ f : [0,2\pi] \to \mathbb{R} \mid f \text{ continuous}, \ f(0) = f(2\pi) \}$$

$$\subset C[0,2\pi]$$

<u>Proof:</u> First note that trig. polynomials form algebra because

$$\sin kt \cdot \sin lt = \tfrac{1}{2} \cos(k-l)t - \tfrac{1}{2} \cos(k+l)t$$

etc...

To use Stone-Weierstrass identify

$$C_{per}[0,2\pi] \mathrel{\hat{=}} C(T) \qquad T = \{(x,y) \mid x^2+y^2=1\} \subset \mathbb{R}^2$$

$$t \mapsto (\cos t, \sin t) \qquad T \text{ compact}$$

trig. polynomials in t $\mathrel{\hat{=}}$ polynomials in coordinates x_1, x_2 on T

$\uparrow$

those are dense in $C(T)$ by 14.8. $\square$

<u>14.9. Remark:</u> From the real version 14.4. one gets easily the following complex version of Stone-Weierstraß:

Let K be compact space and

$$C(K, \mathbb{C}) := \{ f : K \to \mathbb{C} \mid f \text{ continuous} \},$$

equipped with the sup-norm

Let $A \subset C(K, \mathbb{C})$ be unital subalgebra, which separates the points of K and satisfies: $f \in A \Rightarrow \bar{f} \in A$

Then A is dense in $C(K, \mathbb{C})$.

(*) essential points:

i) $K = [0,1]$ compact

ii) $A = \{\text{polynomials}\}$ special subalgebra of $C(K)$

$$\left\{\begin{matrix}\text{normed}\\ \text{vector spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{metric}\\ \text{spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{topological}\\ \text{spaces}\end{matrix}\right\}$$

<u>Def.:</u> 1) A <u>metric space</u> (X, d) is a set X together with a <u>metric</u>

$$d : X \times X \to [0, \infty)$$

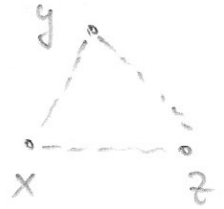
which satisfies:

(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x) \quad \forall x, y \in X$

(iii) triangle inequality:

$$d(x,z) \leq d(x,y) + (y,z) \quad \forall x,y,z \in X$$



[] If $(V, \|\cdot\|)$ is a normed vector space, then $d(x,y) := \|x-y\|$ defines a metric on $X = V$

2) A <u>topology</u> on a set X is a $\tau \subset \mathcal{P}(X) = \{Y \mid Y \subset X\}$ s.th.

i) $\phi, X \in \tau$

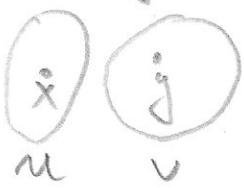
ii) $\mathcal{U}, V \in \tau \Rightarrow \mathcal{U} \cap V \in \tau$

iii) $M \subset \tau \Rightarrow \bigcup_{V \in M} V \in \tau$

[thus τ is closed under finite intersections and arbitrary unions]

Elements $\mathcal{U}$ in τ are called <u>open</u> sets, their complements $\mathcal{U}^c = X \setminus A$ are called <u>closed</u> sets

3) Our topologies are usually <u>Hausdorff</u>, i.e. $\forall x, y \in X$ with $x \neq y$ $\exists \mathcal{U}, V \in \tau$: $x \in \mathcal{U}$, $y \in V$ and $\mathcal{U} \cap V = \phi$



[If (X, d) is a metric space, then

$\mathcal{U}$ open $:\Leftrightarrow \forall x \in \mathcal{U}\ \exists r > 0 : B(x, r) \subset \mathcal{U}$

where

$B(x, r) := \{y \in X \mid d(x, y) < r\}$

defines a topology on X.]

Note: all our relevant properties can be expressed in terms of topology (i.e., in terms of open sets) and sometimes we need topological spaces more general than metric spaces !

<u>Theorem:</u> Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ a function. Then the following are equivalent:

(1) f is continuous, i.e.

$$x_n \to x \implies f(x_n) \to f(x)$$

(2) For each $U \subset Y$ open its preimage

$$f^{-1}(U) \subset X \text{ is open.}$$

(2) serves then as the definition of "continuous" for mappings between arbitrary topological spaces

<u>Theorem (Borel-Lebesgue)</u>: Let X be a metric space and $A \subset X$. TFAE

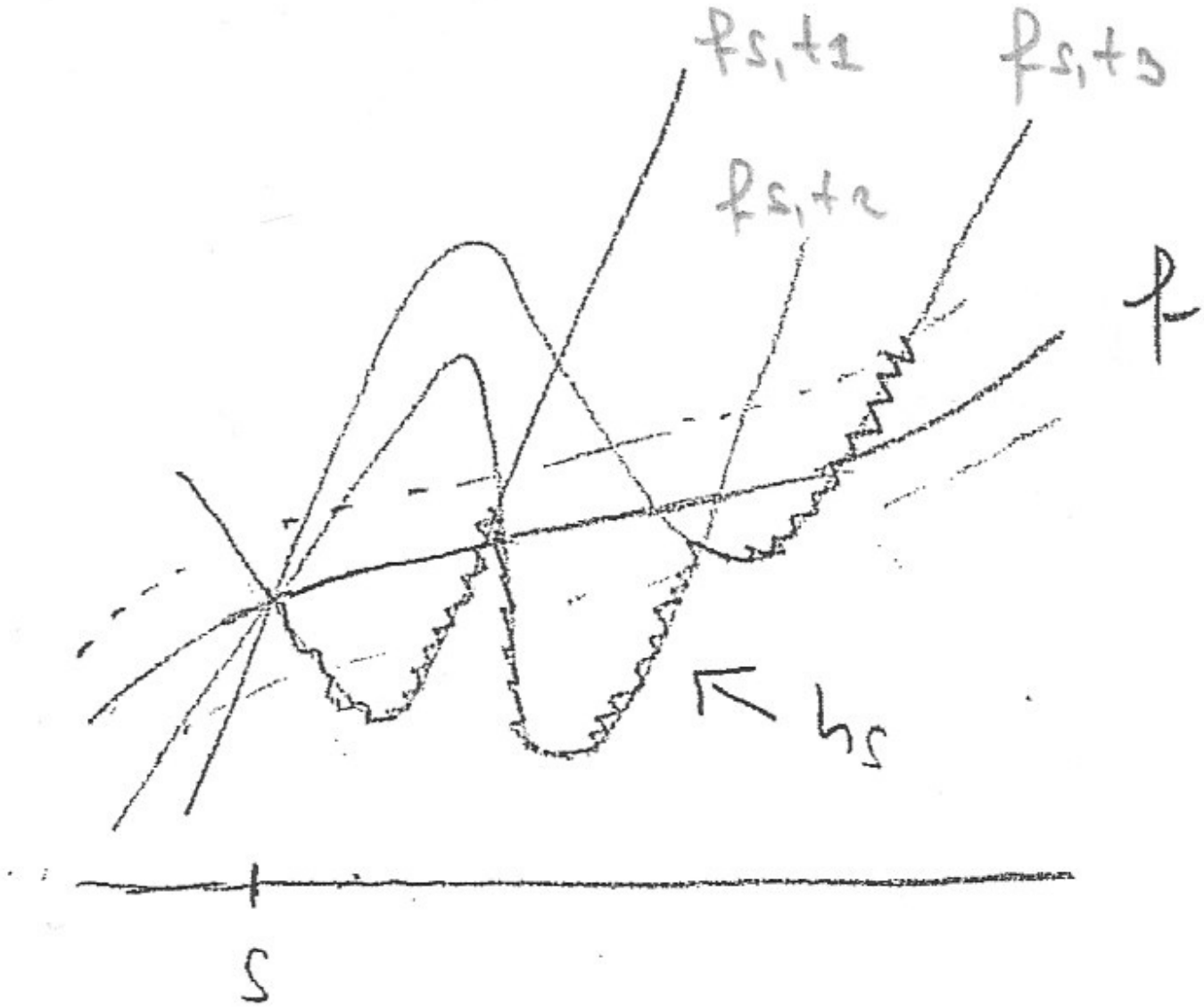
(1) A is (sequentially) compact, i.e., every sequence in A has a convergent subsequence with limit in A.

(2) A is complete and totally bounded, i.e., $\forall \varepsilon > 0$ $\exists$ finitely many $x_1, \dots, x_n \in X$ s.th. $A \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

(3) Every open cover of A has a finite subcover, i.e.

$$\left.\begin{array}{l} (U_i)_{i \in I} \text{ open} \\ A \subset \bigcup_{i \in I} U_i \end{array}\right\} \Rightarrow \begin{array}{l} \exists \text{ finitely many } i_1, \dots, i_n \in I \text{ s.th.} \\ A \subset U_{i_1} \cup \dots \cup U_{i_n} \end{array}$$

(3) is then the "correct" definition of compactness for general topological spaces.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