Exercises 10

Problem 1: Let $W(f)$ ($f \in \mathcal{H}$) be operators satisfying the Weyl relations. Consider $f, g \in \mathcal{H}$ with $f \neq g$ and $\alpha_0, \alpha_f, \alpha_g \in \mathbb{C}$.

- Show that one has then
$$|\alpha_0| \leq \|\alpha_0 1 + \alpha_f W(f) + \alpha_g W(g)\|$$
- Show that one also has
$$|\alpha_f| \leq \|\alpha_0 1 + \alpha_f W(f) + \alpha_g W(g)\|$$

Problem 2: Consider the CCR algebra over the Hilbert space $\mathcal{H}$. Show the following:

(i) We have $\|W(f) - W(g)\| = 2$ for all $f, g \in \mathcal{H}$ with $f \neq g$.

(ii) $CCR(\mathcal{H})$ has trivial center, i.e., the only elements in $CCR(\mathcal{H})$ which commute with all elements in $CCR(\mathcal{H})$ are multiples of the identity: for $A \in CCR(\mathcal{H})$ we have
$$AB = BA \quad \forall B \in CCR(\mathcal{H}) \implies A = \lambda 1, \lambda \in \mathbb{C}.$$

Problem 3: 1) Let A be a simple infinite-dimensional C^*-algebra and $\pi : A \to B(\mathcal{H})$ be a representation. Can the range of π contain non-zero compact operators?

2) Let $\mathcal{H}$ be a finite dimensional Hilbert space. For the Stone-von Neumann uniqueness theorem we contructed in any representation of $CCR(\mathcal{H})$ a projection operator onto the vacuum. Can this projection operator be constructed abstractly in the $CCR(\mathcal{H})$ C^*-algebra?