Exercises 3

1) Show that the spectrum of a general unbounded operator is always closed. Can it be empty? Can it be an unbounded set?

2) Let T be an unbounded operator on $\mathcal{H}$ and assume that the point spectrum is so "big" that we find an orthonormal basis $(x_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ which consists of eigenvectors, i.e., $Tx_k = \lambda_k x_k$ for each k. Show that the spectrum of T lies then in the closure of the point spectrum. (Note that by the previous exercise this means that it is actually equal to the closure of the point spectrum; note also that it is possible that the point spectrum is not closed.)

3) Let $T = i\frac{d}{dt}$ be the closed derivative operator on $L^2([0,1]$, with adjoint T^* and selfadjoint extensions T_α, for each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Recall their domains

$$D(T) = \{f \in \mathcal{H} \mid f \text{ is AC}, f' \in \mathcal{H}, f(0) = 0 = f(1)\}$$
$$D(T_\alpha) = \{f \in \mathcal{H} \mid f \text{ is AC}, f' \in \mathcal{H}, f(1) = \alpha f(0)\}$$
$$D(T^*) = \{f \in \mathcal{H} \mid f \text{ is AC}, f' \in \mathcal{H}, (\text{no boundary conditions})\}$$

Calculate the spectrum of all those operators, and specify also the type of the spectrum (point, continuous, residual).

4) What is the spectrum of the selfadjoint derivative operator $T = i\frac{d}{dt}$ on $L^2(\mathbb{R})$?