Exercises 4

For a selfadjoint operator T on $\mathcal{H}$, with spectral resolution E of the identity, and for any unit vector $x \in \mathcal{H}$ we put

$$\mu_{T,x} := \langle x, E(\cdot)x \rangle.$$

This "spectral measure" is a probability measure on $\sigma(T) \subset \mathbb{R}$, which gives the probability distribution for a measurement of the observable T in the pure state x of the system.

A pure state x corresponds to absolute knowledge about the system. It could also be that one does not have such absolute knowledge, but only statistical information that the system is with probability p_k $(k = 1, \dots, r)$ in the pure state x_k, where $x_1, \dots, x_r$ is an orthonormal system (i.e., unit vectors which are pairwise orthogonal), and where $0 \le p_k \le 1$, $\sum_{k=1}^r p_k = 1$ Then the probability distribution for a measurement of T in such a general state is given by

$$\mu_{T,(p_k),(x_k)} := \sum_{k=1}^r p_k \mu_{T,x_k}$$

.

1) Let A be a selfadjoint operator on $\mathbb{C}^n$ (i.e., a selfajoint matrix) and let $x_1, \dots, x_n$ be an orthonormal basis of eigenvectors of A with $Ax_i = \lambda_i x_i$. Let x be a pure state (i.e., unit vector) with $x = \sum_{i=1}^n \alpha_i x_i$. Determine the spectral measure $\mu_{A,x}$.

2) Consider again the finite dimensional situation $\mathcal{H} = \mathbb{C}^n$ and let $y_1, \dots, y_n$ be an orthonormal basis of $\mathcal{H}$. Then we can consider the state that all y_i have the same probability $p_i = 1/n$ (i.e., we have no idea in which pure state the system is), thus

$$\mu_A = \sum_{i=1}^n \frac{1}{n} \mu_{A,y_i}.$$

Does μ_A depend on the choice of the orthonormal basis $(y_i)_{i=1}^n$? Calculate μ_A for a selfadjoint matrix A. Which information about A is relevant for this?

3) The spectral theorem allows us to define for a selfadjoint operator $T = \int \lambda dE(\lambda)$ its exponential (for fixed $t \in \mathbb{R}$)

$$e^{itT} = \int e^{it\lambda} dE(\lambda).$$

This should agree (in nice cases and appropriate sense) with the definition via power series

$$e^{itT} = \sum_{n=0}^\infty \frac{(it)^n T^n}{n!}.$$

Calculate the latter expansion for the derivation operator $T = i\frac{d}{dt}$ on $L^2(\mathbb{R})$ by applying it to smooth (i.e., infinitely often differentiable) L^2 functions.

And here is, just for fun and general education, a quotation from John Baez:

"Beware: physicists often say self-adjoint when they mean symmetric. This is a result of insufficient education and the following facts:

1) 'sufficiently nice' symmetric operators are self-adjoint, and

2) it's much easier to check if an operator is symmetric than if its self-adjoint - in other words, it can be very difficult to show that a symmetric operator is 'sufficiently nice'.

To check that an operator T is symmetric one must show that $\langle \phi, T\psi \rangle = \langle T\phi, \psi \rangle$ for every $\phi, \psi \in D(T)$; this is often easy to do, and if T is a differential operator it usually amounts to integration by parts. To check that T is self-adjoint one must also show that if $\phi \in \mathcal{H}$ has a $\phi' \in \mathcal{H}$ such that $\langle \phi, T\psi \rangle = \langle \phi', \psi \rangle$, then ϕ lies in $D(T)$. This is usually done indirectly using various theorems, some of which we will discuss later, and many problems in mathematical physics consist of proving that symmetric operators are self-adjoint. This is because the spectral theorem only holds for self-adjoint operators, so self-adjoint operators are much better than merely symmetric operators. We also note term 'hermitian' is also used by mathematicians to mean symmetric, and by physicists to mean self-adjoint, by which they mean symmetric ... we will avoid this term."