Exercises 5

1) Show Theorem 6.1: Let $A \in B(\mathcal{H})$ be a bounded selfadjoint operator and define, for all $t \in \mathbb{R}$, $U_t := e^{itA}$ via power series expansion as

$$U_t := \sum_{n=0}^{\infty} \frac{(it)^n}{n!} A^n.$$

Check (if you have never done so) that the series converges indeed to a bounded operator; think also about whether the so-defined exponential function via power series is the same as we would get by definition via functional calculus.

Now show the following:
(i) U_t is unitary for all $t \in \mathbb{R}$.
(ii) $U_0 = 1$.
(iii) $U_t U_s = U_{t+s}$ for all $s, t \in \mathbb{R}$.
(iv) $\lim_{t\to 0} \|U_t - 1\| = 0$.

2) Now let's do the unbounded version of this, i.e., Theorem 6.3: Let T be an unbounded selfadjoint operator and define $U_t := e^{itT}$, for all $t \in \mathbb{R}$, via functional calculus, i.e.,

$$U_t = \int e^{it\lambda} dE(\lambda), \qquad \text{if} \qquad T = \int \lambda dE(\lambda).$$

(The definition of the exponential function via power series does not necessarily make sense any more for unbounded operators.) Now show the following (by using properties of functional calculus):
(i) U_t is unitary for all $t \in \mathbb{R}$.
(ii) $U_0 = 1$.
(iii) $U_t U_s = U_{t+s}$ for all $s, t \in \mathbb{R}$.
(iv) $\lim_{t\to 0} \|U_t x - x\| = 0$ for all $x \in \mathcal{H}$.

3) Let us try to see what Stone's Theorem means for the derivation operator. When talking about the selfadjointness of this operator we saw that we should distinguish the cases $L^2(\mathbb{R})$ and $L^2(0,1)$.

a) Consider the selfadjoint derivation operator $T = i\frac{d}{dt}$ on the Hilbert space $L^2(\mathbb{R})$. Recall the precise definition of its domain

$$D(T) = \{f \in L^2(\mathbb{R}) \mid f \text{ is AC}, f' \in L^2(\mathbb{R})\}.$$

We checked already last time that the corresponding U_t is given by the shift by amount t, i.e.,

$$(U_t f)(s) = f(s-t).$$

In the proof of Stone's Theorem we used as domain for the generator the set of all vectors f for which the limit

$$\lim_{t\to 0} \frac{U_t f - f}{t}$$

exists. If everything fits together nicely this should be the same as $D(T)$. Try to convince yourself (and maybe also me) that this is indeed the case.

b) Consider now the situation on $L^2(0,1)$. There we had to choose the right boundary conditions, i.e., for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$, we had a selfadjoint extension T_α with domain

$$D(T_\alpha) = \{f \in L^2(0,1) \mid f \text{ is AC}, f' \in L^2(0,1), f(1) = \alpha f(0)\}.$$

So, for each such α we should get a corresponding unitary group $(U_t^\alpha)_{t\in\mathbb{R}}$. Calculate those! Go also the other way, by starting from the U_t^α and calculate from those their generator. The main emphasis here is, of course, to see how the boundary conditions in T_α correspond to conditions for the U_t^α.