Exercises 6

Problem 1: Consider the Schrödinger representation of CCR in the form $[a, a^*] = 1$, with its vacuum vector Ω, i.e., $a\Omega = 0$. We make now, for some real $\lambda, \mu \in \mathbb{R}$, a so-called Bogoliubov transformation by putting

$$b := \lambda a - \mu a^*, \qquad \text{and thus} \qquad b^* = \lambda a^* - \mu a.$$

(i) For which λ, μ does this also give a representation of the CCR, in the form $[b, b^*] = 1$.

(ii) Since we believe that this new representation is equivalent to the Schrödinger representation, there should also be a vacuum vector ω for b, i.e., a vector ω in the Hilbert space such that $b\omega = 0$. Find such an ω!

Problem 2: One has the following

Proposition: Let $P : D(P) \to \mathcal{H}$ and $Q : D(Q) \to \mathcal{H}$ both be selfadjoint and put

$$U_t := e^{itP} \qquad \text{and} \qquad V_t := e^{itQ}.$$

Then the following statements are equivalent:

(a) We have for all $t \in \mathbb{R}$ that $U_t D(Q) \subset D(Q)$ and that on $D(Q)$ the relation

$$U_t Q U_{-t} = Q + t1$$

holds.

(b) We have for all $s, t \in \mathbb{R}$:

$$U_t V_s = e^{ist} V_s U_t.$$

We used property (b) to define the Weyl relations.

(i) Show that property (a) is satisfied for the Schrödinger representation.

(ii) Prove the proposition, i.e., show that (a) and (b) are equivalent in general.

Problem 3: Let (U_t, V_s) be a representation of the Weyl relations. We put, for $s, t \in \mathbb{R}$

$$W(s,t) := e^{-\frac{1}{2}ist} U_s V_t = e^{\frac{1}{2}ist} V_t U_s.$$

(i) Show that in terms of the $W(s,t)$ the Weyl relations are equivalent to

$$W(s_1, t_1) W(s_2, t_2) = W(s_1 + s_2, t_1 + t_2) e^{\frac{1}{2} i (s_1 t_2 - s_2 t_1)}.$$

If we write this in terms of complex variables $z = s + it$, then it takes on the even nicer form

$$W(z_1) W(z_2) = W(z_1 + z_2) e^{\frac{1}{2} \Im(\bar{z}_1 z_2)}.$$

(ii) Show that all $W(s,t)$ are unitary.