Exercises 7

Problem 1: In the Schrödinger representation the operator $P = W_h$ (the clash of notation with the momentum operator is really a bit unfortunate, but hopefully not too confusing) is acting as

$$(Pf)(x) = \frac{1}{2\pi}\int e^{\frac{1}{2}ist}e^{itx}f(x+s)e^{-\frac{1}{4}(s^2+t^2)}dsdt.$$

In class I claimed that this is indeed the projection onto the vacuum, i.e.,

$$Pf = \langle \Omega, f\rangle \Omega, \qquad \text{where} \qquad \Omega(x) = \frac{1}{\pi^{1/4}}e^{-x^2/2}.$$

Show that this is true.

Problem 2: We will address here Heisenberg's uncertainty principle. For a selfadjoint operator T and a unit vector $\psi \in D(T^2)$ we define the *standard deviation* of T in the state ψ by

$$\Delta(T) := \left(\langle \psi, T^2\psi\rangle - \langle \psi, T\psi\rangle^2\right)^{1/2}.$$

(Usually the dependency of $\Delta(T)$ on ψ is supressed in the notation.) This $\Delta(T)$ is the standard deviation of the probability distribution for the measurment of the observable T in the state ψ.

(i) Prove the following *Uncertainty Principle*: Let A and B be self-adjoint operators, and suppose

$$\psi \in D(A^2) \cap D(B^2) \cap D(AB) \cap D(BA)$$

is a unit vector. Then

$$\Delta(A)\Delta(B) \geq \frac{1}{2}|\langle \psi, [A,B]\psi\rangle|.$$

(ii) What does this give for the momentum and position operator in the Schrödinger representation? Find a vector ψ in this representation such that equality is attained.

Problem 3: Show the following corollary of the Stone-von Neumann uniqueness theorem: Let (U_t, V_s) be a represention of the Weyl relations on a separable Hilbert space $\mathcal{H}$. Let P be the generator of U_t, and Q the generator of V_s. Then there is a dense domain $D \subset \mathcal{H}$ so that

(a) $P : D \to D$, $Q : D \to D$
(b) $PQ\psi - QP\psi = -i\psi$ for all $\psi \in D$
(c) P and Q are essentially selfafjoint on D.