Exercises 8

Problem 1: Let $T \in B(\mathcal{H})$ be a bounded operator and

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \Gamma_+^n(T)$$

its second quantization on the symmetric Fock space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_+^{\otimes n}$$

(i) Prove that

$$\|\Gamma_+^n(T)\| = \|T\| \qquad \text{for all } n \geq 1.$$

(ii) Let $T = U$ be a unitary operator on $\mathcal{H}$. What can you then say about

$$\Gamma_+(U)A(f)\Gamma_+(U)^* \qquad \text{and} \qquad \Gamma_+(U)A^+(f)\Gamma_(U)^*,$$

where $A(f)$ and $A^+(f)$ are the annihilation and creations operators, respectively, for $f \in \mathcal{H}$.

Problem 2: Let P_1, Q_1 be a representation of CCR on $\mathcal{H}_1$ and let P_2, Q_2 be a representation of CCR on $\mathcal{H}_2$. Check that then

$$P_1 \otimes 1, Q_1 \otimes 1, 1 \otimes P_2, 1 \otimes Q_2 \qquad \text{on } \mathcal{H}_1 \otimes \mathcal{H}_2$$

provide a representation of the CRR for two degrees of freedom. Do this formally for the momentum and position operators, and also more rigorous for the corresponding Weyl relations.

Problem 3:

(i) On the symmetric Fock space $\mathcal{F}_+(\mathcal{H})$ consider, for $f \in \mathcal{H}$, the unbounded selfadjoint operator $Q(f) := A(f) + A^+(f)$. As we know any vector of norm 1 induces a probability measure for a selfadjoint operator. What is this probability measure for $Q(f)$ for the vacuum vector

(ii) Consider now two such operators $Q(f)$ and $Q(g)$ for which $\langle f, g\rangle = 0$. Then $Q(f)$ and $Q(g)$ commute and according to the extension of the spectral theorem to commuting operators there is now a probability measure on $\mathbb{R}^2$ for the collection $(Q(f), Q(g))$ with respect to the vacuum vector. Describe this probability measure.

[For this spectral theorem for collections see page 13 of the lecture notes of John Baez; we had some small exchange on this in a teams discussion. This version is for the bounded case, but should also be true for nice unbounded cases, as we can assume here. More concretely, we are looking here for a probability measure in $\mathbb{R}^2$ which has the same moments as our two operators have with respect to $\langle\Omega, \cdot\Omega\rangle$.]