Exercises 9

Problem 1: Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F}_+(\mathcal{H})$ the corresponding symmetric Fock space. For $f \in \mathcal{H}$ we consider the *coherent vector*

$$\varepsilon(f) := \sum_{n=0} \frac{f^{\otimes n}}{\sqrt{n!}} = \Omega + f + \frac{1}{\sqrt{2}} f \otimes f + \frac{1}{\sqrt{6}} f \otimes f \otimes f + \cdots \in \mathcal{F}_+(\mathcal{H}).$$

Show the following

(i) For $f, g \in \mathcal{H}$ we have

$$\langle \varepsilon(f), \varepsilon(g) \rangle = e^{\langle f, g \rangle}$$

(ii) The vector space of finite linear combinations of coherent vectors is dense in $\mathcal{F}_+(\mathcal{H})$.

(iii) Every finite family of coherent vectors is linearly independent.

(iv) Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces. Then there exists a unique unitary isomorphism

$$U : \mathcal{F}_+(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathcal{F}_+(\mathcal{H}_1) \otimes \mathcal{F}_+(\mathcal{H}_2)$$
$$\varepsilon(f \oplus g) \mapsto \varepsilon(f) \otimes \varepsilon(g)$$

Problem 2: Let $(\mathcal{H}_i)_{i \in \mathbb{N}}$ be a sequence of Hilbert spaces and $e_i \in \mathcal{H}_i$ be a corresponding sequence of unit vectors. Let, for each $i \in \mathbb{N}$, $\Lambda_i \subset \mathcal{H}_i$ be an orthonormal basis of $\mathcal{H}_i$ which contains e_i. Describe, in terms of the $(\Lambda_i)_{i \in \mathbb{N}}$, an orthonormal basis for the infinite tensor product

$$\bigotimes_{i \in \mathbb{N}} (\mathcal{H}_i, e_i).$$

Problem 3: Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. Show that, in the case where the α_n are all real and $\alpha_n > 1$ for all n, the following two statements are equivalent:

$$\sum_{n \in \mathbb{N}} |\alpha_n - 1| < \infty \qquad \text{and} \qquad \prod_{n \in \mathbb{N}} \alpha_n \text{ converges.}$$

How is the situation for general α_n?