# A. Lebesgue-Stieltjes Integral and Absolutely Continuous Functions

This gives the basic measure theoretic prerequisites of the course. We want to integrate on $\mathbb{R}$ functions $f$ against a measure $\mu$

$$\int f(t)\, d\mu(t).$$

We consider, for convenience, only finite measures.

On $\mathbb{R}$ a finite measure $\mu$ can be encoded by its distribution function $h$,

$$h(t) := \mu((-\infty, t])$$

and then we write $d\,h(t)$ instead of $d\mu(t)$. Integrals of the form

$$\int f(t)\, d\,h(t)$$

are called Stieltjes integrals.

For $f$ $\left\{\begin{matrix}\text{continuous}\\ \text{measurable}\end{matrix}\right\}$ one has $\left\{\begin{matrix}\text{Riemann}\\ \text{Lebesgue}\end{matrix}\right\}$ version of the theory.

<u>A.1. General setting</u>: We consider in the following a function

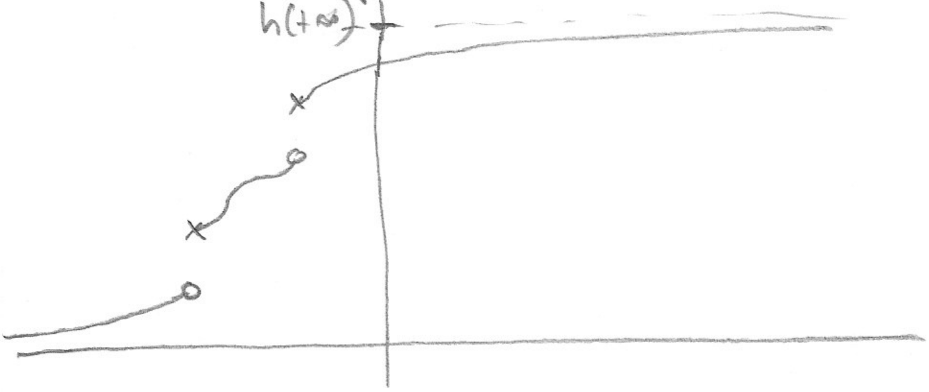
$h : \mathbb{R} \to \mathbb{R}$

with the properties:

i) $\lim_{t\to-\infty} h(t) = 0$, $\lim_{t\to+1} h(t) < \infty$

ii) $h$ is increasing

iii) $h$ is right continuous



<u>A.2. Theorem and Definition</u>: Let $f$ be a continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. Then the limit

$$\int f(t)\, d h(t) := \underbrace{\lim \sum_i}_{\sup_i |t_{i+1}-t_i| \searrow 0} f(t_i)\left[h(t_{i+1}) - h(t_i)\right]$$

exists and is called <u>Riemann-Stieltjes integral</u> of $f$ w.r.t. $dh$

The class of Riemann integrable functions is too small; in particular, it does not fit well with pointwise convergence of functions

⇝ extension to measurable functions

<u>A.3. "Definition"</u>: <u>Measurable</u> (Borel = Baire) functions $\mathbb{R} \to \mathbb{R}$ are the smallest class of functions, which contains all continuous functions and which is closed under pointwise convergence.

<u>A.4. Remark</u>: Each "concretely constructed" function is measurable. One needs the axiom of choice to show the existence of non-measurable functions!

<u>A.5. "Definitions"</u>: 1) The definition of $\int f(t)\, dh(t)$ can be extended to measurable functions:

$$\left.\begin{matrix} f \text{ measurable} \\ f \geq 0 \end{matrix}\right\} \Rightarrow \begin{matrix} \int f(t)\, dh(t) \in \mathbb{R}_0^+ \cup \{+\infty\} \\ \text{is always defined and is} \end{matrix}$$

called <u>Lebesgue-Stieltjes integral</u>

2) $f : \mathbb{R} \to \mathbb{C}$ measurable is called

<u>integrable</u> $:\iff \int |f(t)| \, d\lambda(t) < \infty$

$L^1(\mathbb{R}, d\lambda) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ integrable} \}$

More general, for $p \in \mathbb{R}$, with $1 \leq p < \infty$:

$$L^p(\mathbb{R}, d\lambda) := \{ f : \mathbb{R} \to \mathbb{C} \text{ measurable} \mid \int |f(t)|^p \, d\lambda(t) < \infty \}$$

<u>A.6. Remarks:</u> 1) For $f \in L^1$, the integral

$$\underbrace{\int f(t) d\lambda(t) = \int \operatorname{Re} f(t) d\lambda(t) + i \cdot \int \operatorname{Im} f(t) d\lambda(t)}_{\in \mathbb{C}}$$

is well-defined and one has

$$\left| \int f(t) d\lambda(t) \right| \leq \int |f(t)| \, d\lambda(t)$$

2) $L^2$ is a Hilbert space with

inner product

$$\langle f, g \rangle := \int \overline{f(t)} \, g(t) \, d\lambda(t)$$

[note: one identifies two fcts, if they agree almost everywhere w.r.t. $d\lambda$ ]

The main workhorses for concrete integration are the convergence theorems.

## A.7. Monotone Convergence Theorem (Beppo Levi):

Let $f_n$ be real-valued measurable functions with

$$0 \leq f_1(t) \leq f_2(t) \leq \dots \quad \forall t \in \mathbb{R}$$

and put

$$f(t) := \lim_{n \to \infty} f_n(t) \in [0, \infty]$$

Then we have

$$\int f(t)\, dh(t) = \lim_{n \to \infty} \int f_n(t)\, dh(t)$$

(with possibly $\infty = \infty$)

## A.8. Dominated Convergence Theorem (Lebesgue):

Let $(f_n)_{n \in \mathbb{N}}$ be sequence of measurable functions which converge pointwise. Put

$$f(t) := \lim_{n \to \infty} f_n(t).$$

Assume that there exists $g \in L^1(\mathbb{R}, dh)$ such that

$$|f_n(t)| \le g(t) \quad \forall t \in \mathbb{R} \ \forall n \in \mathbb{N}.$$

Then $f \in L^1(\mathbb{R}, dh)$ and we have

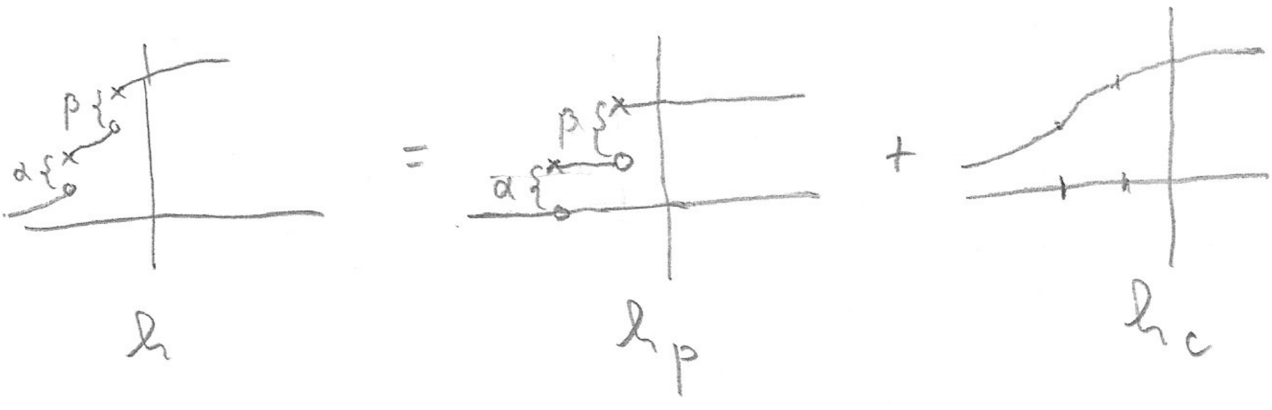
$$\int f(t)\, dh(t) = \lim_{n \to \infty} \int f_n(t)\, dh(t).$$

A.9. Remark: Our functions $h$ as in A.1. can always be uniquely decomposed as

$$h = \underset{\substack{\uparrow \\ \text{atoms} \\ \text{point measure}}}{h_p} + \underset{\substack{\uparrow \\ \text{continuous}}}{h_c}$$

where

- $h_p$ is piecewise constant
- $h_c$ is continuous



For $h_p = \sum_i \mathbb{1}_{[t_i, t_{i+1})} \cdot \alpha_i$ the corresponding integral is given by

$$\int f(t)\, d h_p(t) = \sum_i f(t_i) \cdot (\alpha_{i+1} - \alpha_i)$$

For continuous $h_c$ one might hope to have

$$d h_c(t) = h_c'(t)\, dt$$

and thus reduce $\int f(t)\, d h_c(t)$ to an ordinary "calculus" integral

$$\int f(t) \cdot h_c'(t)\, dt$$

with respect to Lebesgue measure. However, this is only true for absolutely continuous functions!

If we want

$$\int f(t)\, d h_c(t) = \int f(t)\, h_c'(t)\, dt$$

then we need

i) $h_c'$ must be defined, at least almost everywhere w.r.t. Lebesgue measure $dt$

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ii) for $f = 1_{[a,b]}$ we have

$$\int f(t)\, d h_c(t) = \int_a^b d h_c(t) = h_c(b) - h_c(a)$$

and

$$\int f(t)\, h_c'(t)\, dt = \int_a^b h_c'(t)\, dt$$

thus we need for $h_c$ the "fundamental theorem of calculus"

$$\int_a^b h_c'(t)\, dt = h_c(b) - h_c(a)$$

i) is always true (for increasing functions), by a theorem of Lebesgue

ii) is not always true, one needs stronger condition than continuity, $h_c$ must be absolutely continuous

**A.10. Def.:** 1) A set $N \subset \mathbb{R}$ is a **null set** or of **measure zero** (w.r.t. Lebesgue measure), if for each $\varepsilon > 0$ there are countably many intervals $([s_i, t_i])_{i \in \mathbb{N}}$ s.th. $N \subset \bigcup_{i \in \mathbb{N}} [s_i, t_i]$ and

$$\sum_{i \in \mathbb{N}} |t_i - s_i| < \varepsilon$$

2) A property $P$ for real numbers holds **almost everywhere** (a.e.), if $\{t \in \mathbb{R} \mid P \text{ does not hold for } t\}$ is a null set.

**A.11. Remarks:** 1) In particular,

$f = g$ a.e. $\Leftrightarrow$ $\{t \in \mathbb{R} : f(t) \neq g(t)\}$ has measure zero

2) All countable sets are null sets, but there exist also uncountable null sets.

*A.12. Def.:* A function $g:[a,b]\to\mathbb{C}$ is *absolutely continuous* (AC) if

$\forall \varepsilon>0\ \exists \delta>0$ s.th.

$$\left.\begin{array}{l} [s_1,t_1],\ldots,[s_n,t_n] \\ \text{disjoint subintervals} \\ \text{of } [a,b] \text{ with} \\ \sum_{i=1}^{n} |t_i-s_i|<\delta \end{array}\right\} \Rightarrow \sum |g(t_i)-g(s_i)|<\varepsilon$$

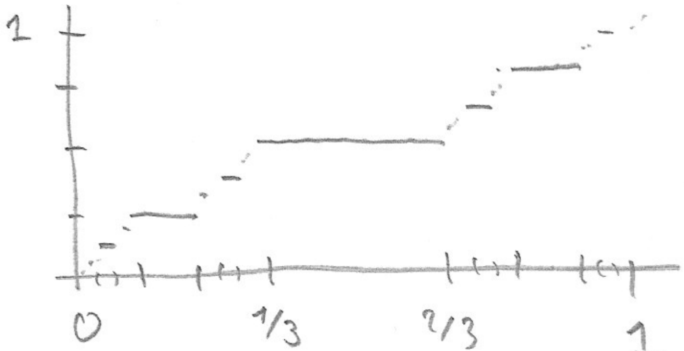
*A.13 Remarks:* 1) AC $\Rightarrow$ uniformly continuous

$\Updownarrow$ on compact set

continuous

2) There exist continuous functions which are not AC.

Example: Cantor fct ("devil's staircase")



This $g$ is *singular*, i.e.: ∘ $g$ continuous
∘ $g$ non-constant
∘ $g'=0$ almost everywhere

hence

$$0=\int_0^t g'(s)\,ds \neq g(t)$$

## A.14. Fundamental Theorem of Calculus:

1) Let $g : [a,b] \to \mathbb{C}$ be $AC$. Then $g$ is differentiable almost everywhere, $g' \in L^1[a,b]$ (i.e. $\int_a^b |g'(s)|\,ds < \infty$) and

$$g(t) = g(a) + \int_a^t g'(s)\,ds.$$

2) Let $f \in L^1[a,b]$ and put

$$g(t) := \int_a^t f(s)\,ds$$

Then $g$ is $AC$ and $g'(t) = f(t)$ a.e.

**A.15. Remark:** Now we can refine the decomposition from A.9. to: Every function $h$ as in A.1 can uniquely be decomposed as

$$h = \underset{\underset{\text{atomic}}{\uparrow}}{h_p} + \underset{\underset{AC}{\uparrow}}{h_{ac}} + \underset{\underset{\text{singular}}{\uparrow}}{h_s},$$

giving a decomposition of the integral:

$$\int f\,dh = \underbrace{\int f\,dh_p}_{\text{series}} + \underbrace{\int f\,dh_{ac}}_{= \int f\,h'_{ac}\,dt \atop \text{"calculus"}} + \underbrace{\int f\,dh_s}_{\text{this better be} \atop \text{absent!}}$$