1. Hilbert Spaces

1.1. Def.: A <u>Hilbert space</u> (HS) $\mathcal{H}$ is a complex vector space, i.e.

$$\left.\begin{array}{l} x, y \in \mathcal{H} \\ \lambda \in \mathbb{C} \end{array}\right\} \Rightarrow \begin{array}{l} x + y \in \mathcal{H} \\ \lambda x \in \mathcal{H}, \end{array}$$

which is equipped with an inner product

$$\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

which satisfies (for all $x, y, z \in \mathcal{H}$)

a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

b) $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle \qquad (\lambda \in \mathbb{C})$

c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

d) $\langle x, x \rangle \geq 0 \quad \forall x \in \mathcal{H}$

$\quad$ "$=$" $\Longleftrightarrow x = 0$

such that $\mathcal{H}$ is complete with respect to the norm $\|x\| := \sqrt{\langle x, x \rangle}$

[complete: every Cauchy sequence in $\mathcal{H}$ converges to a limit in $\mathcal{H}$]

1.2. Examples: 1) $\mathcal{H} = \mathbb{C}^m$

$x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$

$$\langle x, y \rangle := \sum_{n=1}^{m} \bar{x}_n y_n$$

2) $\mathcal{H} = \ell_2 = \{ (x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{C}, \ \sum_{n=1}^{\infty} |x_n|^2 < \infty \}$

$$\langle (x_n), (y_n) \rangle := \sum_{n=1}^{\infty} \bar{x}_n y_n$$

3) $\mathcal{H} = L^2(\mathbb{R}, \mu)$

$x \mathrel{\hat{=}} f \in L^2(\mathbb{R}, \mu)$, i.e.

f measurable with $\int |f(t)|^2 \, d\mu(t) < \infty$

$$\langle f, g \rangle := \int \bar{f}(t) \, g(t) \, d\mu(t)$$

note: $f, g \in L^2 \Rightarrow f \cdot g \in L^1$

i.e. $\left.\begin{array}{l} \langle f, f \rangle < \infty \\ \langle g, g \rangle < \infty \end{array}\right\} \Rightarrow \langle f, g \rangle < \infty$

1.3. Theorem (Cauchy-Schwarz Inequality):

Let $\mathcal{H}$ be HS. Then we have for all $x, y \in \mathcal{H}$:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

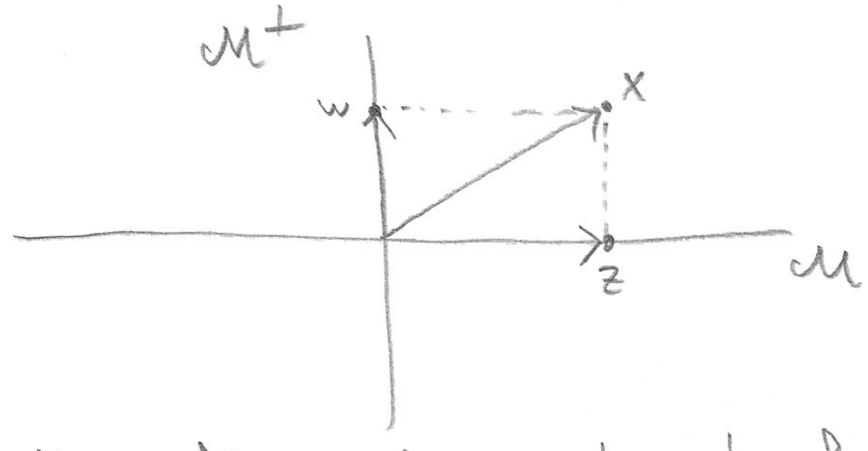
"$=$" $\iff$ x, y are linearly dependent

Proof: $\langle x+\lambda y, x+\lambda y\rangle \geq 0 \quad \forall \lambda \in \mathbb{C}$

choosing $\lambda := \frac{\langle y, x\rangle}{\langle y, y\rangle}$ gives assertion □

1.4. Notation: Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$ (i.e. $\mathcal{M}$ is itself a Hilbert space). Then we put

$$\mathcal{M}^\perp := \{x \in \mathcal{H} : \langle x, y\rangle = 0 \quad \forall y \in \mathcal{M}\}$$



$x = z + w$

$z \in \mathcal{M}$

$w \in \mathcal{M}^\perp$

$\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$.

<u>1.5. Theorem:</u> Let $\mathcal{H}$ be a HS and $\mathcal{M}$ a sub-HS. Then any $x \in \mathcal{H}$ can uniquely be written as

$x = z + w$ with $z \in \mathcal{M}$, $w \in \mathcal{M}^\perp$

We write then

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$$

<u>1.6. Notation:</u> We denote by

$$\mathcal{H}^* := \{ \xi : \mathcal{H} \to \mathbb{C} \mid \xi \text{ linear, continuous} \}$$

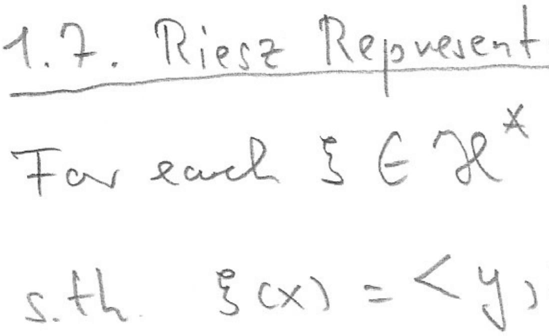
the <u>dual space</u> of $\mathcal{H}$.

<u>1.7. Riesz Representation Theorem:</u> Let $\mathcal{H}$ be HS. For each $\xi \in \mathcal{H}^*$ there is a unique $y \in \mathcal{H}$ s.th. $\xi(x) = \langle y, x \rangle \quad \forall x \in \mathcal{H}$

i.e.: $\mathcal{H}$ is its own dual space

<u>Proof:</u> Put $\mathcal{N} := \ker \xi = \{ z \in \mathcal{H} \mid \xi(z) = 0 \}$

Then $\operatorname{codim} \mathcal{N} = \dim \mathcal{N}^\perp = 1$ $\left(\begin{smallmatrix} \text{or } 0 \text{ if} \\ \xi = 0 \end{smallmatrix} \right)$



Take any $0 \neq x_0 \in \mathcal{N}^\perp$ and normalize it, to get wanted

$$y = \frac{\overline{\xi(x_0)}}{\|x_0\|^2} x_0$$

□

1.8. Remark: This corresponds to Dirac's bra-ket notation in physics:

$$x \,\hat{=}\, |x\rangle \quad \text{ket}$$

$$\xi \,\hat{=}\, \langle y| \quad \text{bra}$$

$$\xi(x) \,\hat{=}\, \underset{\text{bra ket}}{\langle y|x\rangle}$$

1.9. Def.: A Hilbert space is separable if it has a countable dense subset.

We will only consider separable HS!

1.10. Theorem: Each separable HS has a orthonormal basis, i.e. $\{x_i\}_{i=1,\dots,N}$ (where $N \in \mathbb{N}$ or $N = \infty$) with

- $\langle x_i, x_j\rangle = \delta_{ij} \quad \forall i,j$
- $x = \sum_n \langle x_i, x\rangle x_i$
- $\|x\|^2 = \sum_n |\langle x_i, x\rangle|^2$

Such ONB can be produced by the Gram-Schmidt process.

ONB are not unique, but the size N of such an ONB is.

N is called the <u>dimension</u> of $\mathcal{H}$,

$\dim \mathcal{H} = N$

<u>1.11. Theorem:</u> Hilbert spaces with the same dimension are isomorphic; i.e. each separable HS $\mathcal{H}$ is isomorphic to

- $\mathbb{C}^N$ if $\dim \mathcal{H} = N < \infty$
- l_2 if $\dim \mathcal{H} = \infty$