10. Infinite tensor product of Hilbert spaces and inequivalence of representations of CCR(∞)

Denote

$CCR(n) \hat{=} CCR$ for n degrees of freedom

We are interested in representations of CCR(∞)!

Consider

$$\left.\begin{array}{l} P_1, Q_1 \text{ on } \mathcal{H}_1 \\ P_2, Q_2 \text{ on } \mathcal{H}_2 \end{array}\right\} \text{representation of } CCR(1)$$

Then

$$\left.\begin{array}{l} P_1 \otimes 1, \; Q_1 \otimes 1 \\ 1 \otimes P_2, \; 1 \otimes Q_2 \end{array}\right\} \text{ on } \mathcal{H}_1 \otimes \mathcal{H}_2$$

give representation of CCR(2)

For example, Schrödinger representation of CCR(n) on $L^2(\mathbb{R}^n)$ is n-fold tensor product of representations of CCR(1) on $L^2(\mathbb{R})$, compare 8.6

<u>Idea</u>: represent CCR(∞) as "infinite" tensor product on $\mathcal{H}^{\otimes\infty}$ of representation of CCR(1) on $\mathcal{H}$

<u>Problem</u>: notion of infinite tensor product is problematic, there are actually many inequivalent ones!

Consider Hilbert spaces $\mathcal{H}_i$, for $i \in \mathbb{N}$. We want to define

$\bigotimes_{i\in\mathbb{N}} \mathcal{H}_i$, i.e. we need

$$(*) \quad \langle x_1 \otimes x_2 \otimes \dots, y_1 \otimes y_2 \otimes \dots \rangle \overset{?}{\doteq} \underbrace{\prod_{i\in\mathbb{N}} \langle x_i, y_i \rangle}_{\text{convergence?}}$$

Instead of allowing the most general convergent situation ("complete" tensor product) we restrict to more special, controllable situations ("incomplete" tensor products)

→ convergence in (*) is clear, if

$\langle x_i, y_i \rangle = 1$ for large i, i.e.

for $x_i = y_i = e_i$ for large i

for fixed $e_i \in \mathcal{H}_i$ with $\|e_i\| = 1$.

10.1. Definition (von Neumann 1939):

Let $\mathcal{H}_i$, $i \in \mathbb{N}$, be Hilbert spaces and $e_i \in \mathcal{H}_i$ for each $i \in \mathbb{N}$ a fixed unit vector. The (incomplete) tensor product $\bigotimes_{i \in \mathbb{N}} (\mathcal{H}_i, e_i)$ is the completion, after factoring out null vectors, of the linear span of

$$\{(x_1, x_2, x_3, \dots) \mid x_i \in \mathcal{H}_i \text{ for all } i \in \mathbb{N} \text{ and } x_i = e_i \text{ for all but finitely many } i\}$$

with respect to inner product (*).

10.2. Rem: Fixing the $(e_i)_{i \in \mathbb{N}}$ gives us a way of embedding $\mathcal{H}^{\otimes n}$ in $\mathcal{H}^{\otimes (n+1)}$

via $\mathcal{H}^{\otimes n} \cong \mathcal{H}^{\otimes n} \otimes e_{n+1} \subset \mathcal{H}^{\otimes (n+1)}$

comp. 9.3(2)

and thus elements

$$x = x_1 \otimes x_2 \otimes x_3 \otimes \ldots \in \bigotimes_{i \in \mathbb{N}} (\mathcal{H}_i, e_i)$$

have to be understood as

$$x = \lim_{n \to \infty} x_1 \otimes \ldots \otimes x_n \otimes e_{n+1} \otimes e_{n+2} \otimes \ldots$$

Suppose that $\|x_i\| = 1 \quad \forall i$, then

the existence of x as the limit

requires that

$$\| x_1 \otimes \ldots \otimes x_n \otimes e_{n+1} \otimes \ldots - x_1 \otimes \ldots \otimes x_m \otimes e_{m+1} \|$$

gets arbitrarily small $(<\varepsilon)$ for sufficiently

large $n, m \quad (\geq N(\varepsilon))$

$$= \| e_{n+1} \otimes \ldots \otimes e_m - x_{n+1} \otimes \ldots \otimes x_m \|$$

$$= 1 + 1 - \prod_{k=n+1}^{m} \langle e_k, x_k \rangle - \prod_{k=n+1}^{m} \langle x_k, e_k \rangle$$

The right condition to ensure this

(and that everything else works) is

$$\sum_{k=1}^{\infty} | \langle e_k, x_k \rangle - 1 | < \infty$$

~~and thus~~

$$\cancel{x = x_1 \otimes x_2 \otimes \dots \overset{?}{=} \lim_{n\to\infty} x_1 \otimes \dots \otimes x_n \otimes e_{n+1} \otimes e_{n+2} \otimes \dots}$$

10.3. Theorem (von Neumann): For two sequences $(e_i)_{i\in\mathbb{N}}$ and $(f_i)_{i\in\mathbb{N}}$ of unit vectors TFAE.

a) $\bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, e_i)$ can be identified with $\bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, f_i)$ i.e., there is a unitary $\mathcal{U}: \bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, e_i) \to \bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, f_i)$ such that for all $n\in\mathbb{N}$ the diagram

$$\begin{tikzcd}
 & \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n \arrow[dl] \arrow[dr] & \\
\bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, e_i) \arrow[rr, "\mathcal{U}"] & & \bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, f_i)
\end{tikzcd}$$

commutes, where the vertical maps are the canonical embeddings, i.e.

$$x_1 \otimes \dots \otimes x_n \longmapsto x_1 \otimes \dots \otimes x_n \otimes e_{n+1} \otimes e_{n+2} \otimes \dots$$

$$\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n \longrightarrow \bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, e_i)$$

b) $\sum_{i=1}^{\infty} | \langle e_i, f_i \rangle - 1 | < \infty$

"<u>Proof</u>": b) $\Rightarrow$ a)

For example, the element

$$e = e_1 \otimes e_2 \otimes e_3 \otimes \dots \in \bigotimes_{i \in \mathbb{N}} (\mathcal{H}_i, e_i)$$

has to appear as

$$\mathcal{U} e = \lim_{n \to \infty} e_1 \otimes \dots \otimes e_n \otimes f_{n+1} \otimes f_{n+2} \otimes \dots$$

$$\in \bigotimes_{i \in \mathbb{N}} (\mathcal{H}_i, f_i)$$

That this limit exists is, by 10.2., ensured by our condition

$$\sum_{i=1}^{\infty} | \langle e_i, f_i \rangle - 1 | < \infty$$

a) $\Rightarrow$ b) To distinguish different tensor products can be seen via "superselection rules" as in following example

"$\square$"

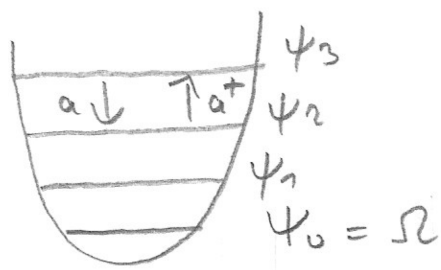
10.4. Example: Inequivalent representations of CCR(∞)

Consider the Schrödinger representation of CCR(1) in the form

$\mathcal{H} = L^2(\mathbb{R})$

$[a, a^+] = 1$ with $a\Omega = 0$

and $\Psi_n := \frac{1}{\sqrt{n!}} a^{+n} \Omega$ ($\Psi_n \hat{=} f_n$ in 7.4)

CCR(1) $\hat{=}$ 

Realize CCR(∞) now as infinite tensor product of this; we can make different choices for this.

We take $\mathcal{H}_i = \mathcal{H}$ for all $i \in \mathbb{N}$

and $e_i \in \mathcal{H}_i = \mathcal{H}$ as

$$e_i = \Psi_n \quad \text{for all } i \in \mathbb{N}$$

for a fixed $n \in \mathbb{N}$

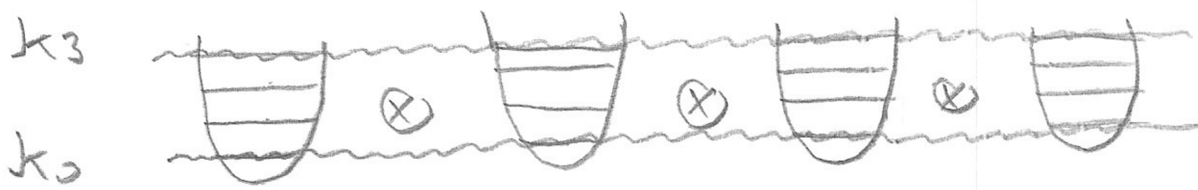
Then we put

$$\mathcal{H}_n := \bigotimes_{i\in\mathbb{N}} (\mathcal{H}_i, e_i) = \bigotimes_{i\in\mathbb{N}} (\mathcal{H}, \Psi_n)$$

and can realise CCR(∞) there via

$$a_i = 1\otimes\cdots\otimes 1\otimes \underset{\substack{\uparrow \\ i\text{-th position}}}{a} \otimes 1\otimes 1\otimes\cdots$$

We claim: representations on $\mathcal{H}_n$ and $\mathcal{H}_m$ are inequivalent for $n \neq m$?



[Our Fock space representation corresponds to representation on $\mathcal{H}_0$]

We distinguish realisations by superselection rules, which take on different values on different $\mathcal{H}_n$.

Consider for CCR(1) the "number operator" a^*a

$$a^*\underbrace{a\,\Psi_n}_{\sqrt{n}\,\Psi_{n-1}} = \sqrt{n}\,\underbrace{a^*\Psi_{n-1}}_{\sqrt{n}\,\Psi_n} = \sqrt{n}\cdot\sqrt{n}\,\Psi_n = n\,\Psi_n$$

For our representation of CCR(∞) on $\mathcal{H}_n$ we consider now

$$N_i := a_i^* a_i \quad \text{number operator on } \mathcal{H}_i$$

and

$$N := \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} N_i \quad \text{averaged total number operator}$$

Note that the limit does not exist algebraically in CCR(∞), but converges in each representation: On $\mathcal{H}_n$ we have for elements x of the form

$$x = \psi_{n_1} \otimes \psi_{n_2} \otimes \dots \otimes \psi_{n_\ell} \otimes \psi_n \otimes \psi_n \otimes \dots$$

that

$$Nx = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (\overbrace{n_1 + n_2 + \dots + n_\ell + n + n + \dots + n}^{k \text{ terms}}) x$$

$$= n\, x$$

This goes over to sums and limits, thus we have

$$Nx = nx \quad \forall x \in \mathcal{H}_n$$

Assume now that representations on $\mathcal{H}_n$ and $\mathcal{H}_m$ are unitarily equivalent, i.e. there is unitary

$$\mathcal{U} : \mathcal{H}_n \to \mathcal{H}_m$$

s.th. $a_i^{(n)} = \mathcal{U}^* a_i^{(m)} \mathcal{U}$

then

$$N_i^{(n)} = \mathcal{U}^* N_i^{(m)} \mathcal{U}$$

and also

$$N^{(n)} = \mathcal{U}^* N^{(m)} \mathcal{U}$$

but then

$$\underbrace{N^{(n)} x}_{n x} = \mathcal{U}^* \underbrace{N^{(m)} \mathcal{U} x}_{m \cdot \mathcal{U} x} = m x \quad \forall x \in \mathcal{H}_n$$

$\Rightarrow n = m$

thus: representations with $n \neq m$ are <u>not</u> unitarily equivalent !

10.5. Conclusion: Choice of a fixed (representation) Hilbert space is in general not adequate.

Instead an algebraic approach is more appropriate

physical system $\hat{=}$ algebra of observables

different physical situations $\hat{=}$ different representations

however: the algebra should not forget that it deals with operators on Hilbert spaces

↳ consider C^*-algebra generated by algebraic relations

Segal 1947

Haag, Kastler 1964