11. The CCR algebra as C^*-algebra of the Weyl relations

Recall our setting for the realization of CCR(∞)

$\mathcal{H}$ Hilbert space

$\mathcal{F}_+(\mathcal{H})$ corresponding symmetric Fock space

$A^+(f)$ creation operator
$A(f)$ annihilation operator $(f \in \mathcal{H})$

Then define

$$\phi(f) := \frac{1}{\sqrt{2}}\left(A^+(f) + A(f)\right)$$

essentially selfadjoint

thus

$$W(f) := e^{i\phi(f)} \quad \text{unitary}$$

note that $f \mapsto A^+(f)$ is linear

but $f \mapsto A(f)$ is anti-linear

and thus

$$\phi(if) = \frac{1}{\sqrt{2}}\left(A^+(if) + A(if)\right)$$

$$= \frac{1}{\sqrt{2}}\, i \left(A^+(f) - A(f)\right)$$

hence, $\phi(f)$ encodes both the position and momentum operators P, Q

The Weyl operators satisfy

symplectic form

$$(*) \begin{cases} \circ\ W(0) = 1 \\ \circ\ W(f) \text{ unitary} \\ \circ\ W(f)^* = W(-f) \qquad \sigma(f,g) = -\sigma(g,f) \\ \circ\ W(f)W(g) = e^{-i\frac{1}{2}\overbrace{\mathrm{Im}\langle f,g\rangle}^{}} W(f+g) \end{cases}$$

Those relations encode the CCR in the Weyl form.

As we have seen there is in general no uniqueness of irreducible representation, thus it is better to divide the investigation of the CCR into two parts:

algebra gives general properties of quantum field theories

representation corresponds then to properties of specific theory

note that quantum theory always lives on Hilbert spaces, hence it's part of the general theory that our algebra is actually an algebra of (bounded) operators on Hilbert spaces, i.e. we are looking on the C^*-algebra.

Crucial for this is that the relations determine the structure of any $*$-representation

<u>11.1. Theorem:</u> Assume that we have operators $W(f)$ on a Hilbert space $\mathcal{H}_1$ and $\tilde{W}(f)$ — " — $\mathcal{H}_2$

such that both $W(f)$, $f \in \mathcal{H}$, and $\tilde{W}(f)$, $f \in \mathcal{H}$, satisfy the relations $(*)$. Then their generated C^*-algebras

$$\mathcal{W} := \overline{\left\{ \begin{matrix} \text{algebra generated by all} \\ W(f),\ f \in \mathcal{H} \end{matrix} \right\}}^{\|\cdot\|} \subset B(\mathcal{H}_1)$$

$$\tilde{\mathcal{W}} := \overline{\left\{ \begin{matrix} \text{———— '' ————} \\ \tilde{W}(f),\ f \in \mathcal{H} \end{matrix} \right\}}^{\|\cdot\|} \subset B(\mathcal{H}_2)$$

are $*$-isomorphic such that

$$W(f) \mapsto \tilde{W}(f) \qquad \forall f \in \mathcal{H}$$

In particular, this means that the norms of the operators are determined by the algebraic relations.

<u>Proof</u>: First note that

algebra generated by all $W(f)$

is just the linear span of all $W(f)$,

since $W(f)W(g) = e^{\cdots} W(f+g)$

So $\mathcal{W}$ consist of limit of elements

$$W = \sum_{\text{finite}} \alpha_f W(f)$$

main point to see is: if $W = 0$, then corresponding $\tilde{W} = \sum \alpha_f \tilde{W}(f)$ must also be zero!

however: we need this also for limits (i.e., infinite sums), thus we must control the norms in these arguments!

so we need a norm control

$$|\alpha_f| \leq C_1 \cdot \| \sum_{g \in I} \alpha_g W(g) \| \quad \forall f \in I$$

just relying on Weyl relations and general facts about operators on Hilbet spaces.

Since Weyl relations allow to move any $W(f)$ to $W(0) = 1$, the crucial part is to separate $W(0)$ from a sum.

Hence, we have to see that

$$|\alpha_0| \leq \| \alpha_0 \cdot 1 + \sum_{f \neq 0} \alpha_f W(f) \|$$

crucial fact:

- conjugation by unitary operators does not change norm, i.e.

$$\| W(g) A W(g)^* \| = \| A \|$$

$\forall$ bounded A

- for the Weyl operators we have

$$W(g)\,W(f)\,W(g)^* = e^{-\frac{1}{2}\sigma(g,f)} \cdot W(g+f)\,W(-g)$$

$$= e^{-\frac{1}{2}\sigma(g,f)} \cdot e^{-\frac{1}{2}\sigma(g+f,-g)}\,W(f)$$

$$= \underbrace{e^{-i\sigma(g,f)}}_{\substack{\text{always } = 1 \text{ if } f = 0 \\ \text{can be made } \neq 1 \text{ if } f \neq 0}}\,W(f)$$

this allows to separate 1 from the other $W(f)$'s ; e.g., to see

$$|\alpha_0| \leq \|\alpha_0 \cdot 1 + \alpha\,W(f)\| \qquad \text{for } f \neq 0$$

we note that

$$\|\alpha_0 \cdot 1 + \alpha\,W(f)\| = \|W(g)\,[\alpha_0 \cdot 1 + \alpha\,W(f)]\,W(g)^*\|$$

$$= \|\alpha_0 \cdot 1 + \alpha\,\underbrace{e^{-i\sigma(g,f)}}_{= e^{-t}}\,W(f)\|$$

where any $t \in \mathbb{R}$ can arise by appropriate choice of g

thus

$$\|\alpha_0 \cdot 1 + \alpha\,W(f)\| = \|\alpha_0 \cdot 1 + e^{-it}\alpha\,W(f)\|$$

$$\forall\, t \in \mathbb{R}$$

$$\Rightarrow \|\alpha_0 \cdot 1\| = \|\alpha_0 \cdot 1 + \frac{1}{2\pi}\int_0^{2\pi} e^{-it}\, dt\, \alpha W(f)\|$$

$$= \|\frac{1}{2\pi}\int_0^{2\pi} [\alpha_0 \cdot 1 + e^{-it}\alpha W(f)]\, dt\|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\|\alpha_0 \cdot 1 + e^{-it}\alpha W(f)\|}_{=\|\alpha_0 \cdot 1 + \alpha W(f)\| \quad \forall t}\, dt$$

$$= \|\alpha_0 \cdot 1 + \alpha W(f)\| \qquad (\text{for all } \alpha \in \mathbb{C},\ f \in \mathcal{H})$$

Case of more summands follows similarly by induction □

<u>11.2. Def.:</u> The C^*-algebra which is uniquely determined by the Weyl relations (*) is called the <u>CCR algebra</u> over $\mathcal{H}$; it is denoted by $CCR(\mathcal{H})$ [or by $CCR(H, \sigma)$ for a general symplectic space (H, σ)].

11.3. Remarks: 1) Theorem 11.1. includes in particular the statements:

i) $CCR(\mathcal{H})$ is a simple C^*-algebra (i.e. contains no non-trivial closed ideals)

ii) A linear mapping $T: \mathcal{H} \to \mathcal{H}$ which preserves the symplectic structure, i.e.

$$\sigma(Tf, Tg) = \sigma(f, g) \quad \forall f, g$$

can be implemented as a $*$-automorphism of $CCR(\mathcal{H})$, i.e., there exists a unique $*$-automorphism

$$\gamma_T: CCR(\mathcal{H}) \to CCR(\mathcal{H}) \quad \text{s.th.}$$

$$\gamma_T(W(f)) = W(Tf)$$

proof: just put

$$\tilde{W}(f) := W(Tf)$$

in 11.1

2) Such mappings T usually correspond to symmetries or evolutions of the considered system

3) In general, in representations of $CCR(\mathcal{H})$ such automorphisms γ_T need not to be unitarily implementable

11.4. Def.: 1) A (unital) C^*-algebra A is a complex algebra with involution $x \mapsto x^*$ and a norm $\|\cdot\|$, s.th.

- A is complete w.r.t. $\|\cdot\|$
- $\|xy\| \leq \|x\| \cdot \|y\|$
- $\|xx^*\| = \|x\|^2$ $\quad \forall x, y \in A$

2) A state ω on a C^*-algebra A is a linear mapping $\omega : A \to \mathbb{C}$ s.th.

- $\omega(1) = 1$
- $\omega(xx^*) \geq 0 \quad \forall x \in A$

3) A representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ s.th. $\pi(1) = 1$

11.5. Theorem (GNS construction): Let ω be a state on a C^*-algebra A. Then there exists a representation of A on a Hilbert space $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{H}$ s.th.

$$\omega(x) = \langle \Omega, \pi(x) \Omega \rangle \quad \forall x \in A$$

GNS = Gelfand – Naimark – Segal

1943 $\qquad$ 1947

11.6. Final words on algebraic standpoint

usual quantum mechanical description on

Hilbert space $\mathcal{H}$

observables : s.a. operators T on $\mathcal{H}$

system : unit vector $\Psi \in \mathcal{H}$

relevant quantity ($\hat{=}$ results of measurements)

$$\omega_\Psi(T) = \langle \Psi, T\Psi \rangle$$

time evolutions or symmetries are described by unitary mappings $\mathcal{U} : \mathcal{H} \to \mathcal{H}$, either by

$$\Psi \mapsto \mathcal{U}\Psi \quad \text{in Schrödinger picture}$$

or by

$$T \mapsto \mathcal{U}^* T \mathcal{U} \quad \text{in Heisenberg picture}$$

Because of

$$\langle \mathcal{U}\Psi, T\mathcal{U}\Psi\rangle = \langle \Psi, \mathcal{U}^* T \mathcal{U}\Psi\rangle$$

$$\omega_{\mathcal{U}\Psi}(T) \qquad\qquad \omega_\Psi(\mathcal{U}^* T \mathcal{U})$$

this describes the same physics

However: in Schrödinger picture $\Psi \mapsto \mathcal{U}\Psi$ only makes sense if we stay in given Hilbert space, whereas $T \mapsto \mathcal{U}^* T \mathcal{U}$ can also be generalized to $*$-automorphism on C^*-algebraic level, which does not need to be unitarily implementable. Hence a more appropriate picture is the C^*-algebraic Heisenberg picture, where

observables $\hat{=}$ s.a. elements of C^*-algebra A

concrete physical system $\hat{=}$ state on A

automorphisms on A can then be rewritten as mappings on state space, but the corresponding representations do not need to be unitarily equivalent.