2. Bounded Operators

In the following, $\mathcal{H}$ is always a (separable) Hilbert space.

2.1. Def.: A linear operator T is a mapping $T: \mathcal{H} \to \mathcal{H}$ which is linear, i.e.

$$T(x+y) = Tx + Ty \qquad \forall x, y \in \mathcal{H}$$
$$T(\lambda x) = \lambda \cdot Tx \qquad \forall \lambda \in \mathbb{C}$$

2.2. Remark: In finite dimensions ($\dim \mathcal{H} < \infty$), any linear operator is automatically continuous. This is not true any more in infinite dimensions!

2.3. Lemma: For a linear operator $T: \mathcal{H} \to \mathcal{H}$ the following are equivalent:

i) T is continuous

ii) T is continuous at 0

iii) T is bounded, i.e.

$$\|T\| := \sup_{\substack{x \in \mathcal{H} \\ \|x\| = 1}} \|Tx\| < \infty$$

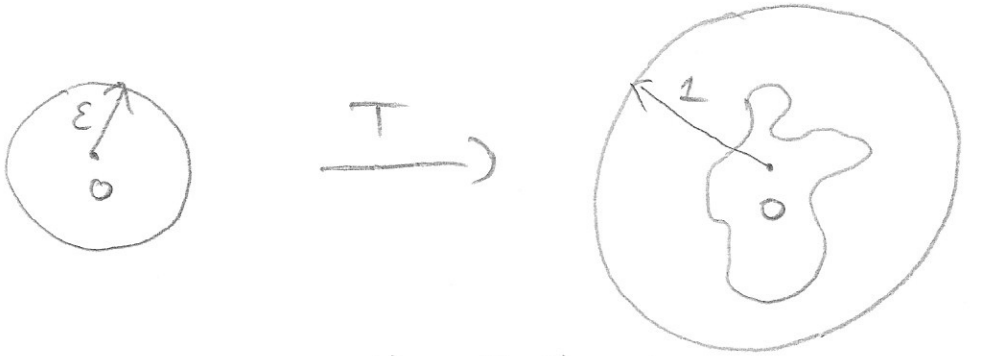
Note that this is saying that $\|T\|$ is the smallest constant s.th.

$$\|Tx\| \leq \|T\| \cdot \|x\| \quad \forall x \in \mathcal{H}$$

<u>Proof</u>: i) ⇒ ii) clea

ii) ⇒ iii) T continuous at 0:

for $\delta = 1$ $\exists \varepsilon > 0 : \|y\| \leq \varepsilon \Rightarrow \|Ty\| \leq 1$



thus for x with $\|x\| = 1$

$$y := \varepsilon x \Rightarrow \|y\| = \varepsilon$$

$$\Rightarrow \underbrace{\|Ty\|}_{\|T(\varepsilon x)\| = \|\varepsilon Tx\| = \varepsilon \|Tx\|} \leq 1$$

$$\Rightarrow \|Tx\| \leq \frac{1}{\varepsilon} \quad \forall x \in \mathcal{H} \text{ with } \|x\| = 1$$

iii) ⇒ i) Assume $x_n \to x$, i.e. $\|x_n - x\| \to 0$

$$\Rightarrow \|Tx - Tx_n\| = \|T(x - x_n)\|$$

$$\leq \|T\| \cdot \underbrace{\|x - x_n\|}_{\to 0}$$

$$\Rightarrow \|Tx - Tx_n\| \to 0 \quad \text{i.e. } Tx_n \to Tx \quad \square$$

2.4. Notation: We call $\|T\|$ the <u>operator norm</u> of T and put

$$B(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} \mid T \text{ linear}, \|T\| < \infty\},$$

the <u>bounded operators</u> on $\mathcal{H}$.

2.5. Theorem: $B(\mathcal{H})$ is a Banach algebra, i.e.

- $B(\mathcal{H})$ is complete w.r.t. $\|\cdot\|$
- $\|T+S\| \leq \|T\| + \|S\|$ ($\|\cdot\|$ is norm)
- $\|T \cdot S\| \leq \|T\| \cdot \|S\|$

2.6. Remark: For $\dim \mathcal{H} = n < \infty$, $B(\mathcal{H})$ can (via choosing an ONB $\mathbb{C}^n$) be identified with $M_n(\mathbb{C})$, the $n \times n$-matrices

For $\dim \mathcal{H} = \infty$, $B(\mathcal{H}) \subsetneq$ formal $\infty \times \infty$ matrices,

but this is not useful; in particular, for $T \hat{=} (t_{ij})_{i,j=1}^{\infty}$ there is no useful criteria in terms of t_{ij} to decide whether $\|T\| < \infty$

2.7. Theorem: Consider $T \in B(\mathcal{H})$.

Then there exists a unique $T^* \in B(\mathcal{H})$ s.th.

$$\langle x, Ty \rangle = \langle T^* x, y \rangle \quad \forall x, y \in \mathcal{H}$$

T^* is called the adjoint of T.

Proof: Let $x \in \mathcal{H}$. How to define $T^* x$?

Consider

$$\xi : \mathcal{H} \to \mathbb{C}$$
$$y \mapsto \langle x, Ty \rangle$$

This is linear and continuous

[$y_n \to y$, i.e. $\|y_n - y\| \to 0$, then:

$$|\xi(y_n) - \xi(y)| = |\xi(y_n - y)|$$
$$= |\langle x, T(y_n - y) \rangle|$$
$$\overset{C-S}{\leq} \|x\| \cdot \underbrace{\|T(y_n - y)\|}_{\leq \|T\| \cdot \underbrace{\|y_n - y\|}_{\to 0}}$$

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$$\Rightarrow \xi(y_n) \to \xi(y)$$

thus: $\xi \in \mathcal{H}^*$

$\overset{\text{Riesz}}{\underset{1.7.}{\Longrightarrow}}$ $\exists\, z \in \mathcal{H}$ s.th.

$$\langle x, Ty\rangle = \xi(y) = \langle z, y\rangle$$

Put $T^*x = z$ and check that

T^* linear and bounded

$\rightsquigarrow T^* \in B(\mathcal{H})$ $\square$

⊛ 2.8. Example: In the finite-dim. setting, $T \rightsquigarrow T^*$ corresponds to taking complex conjugate of matrix

2.9. Theorem: We have for all $T \in B(\mathcal{H})$:

a) $T^{**} = T$

b) $\|T^*\| = \|T\|$

c) $\|TT^*\| = \|T\|^2$ (C^*-condition)

2.10. Def.: Consider $T, P, U, V, N \in B(\mathcal{H})$.

a) T is <u>selfadjoint</u>, if $T = T^*$

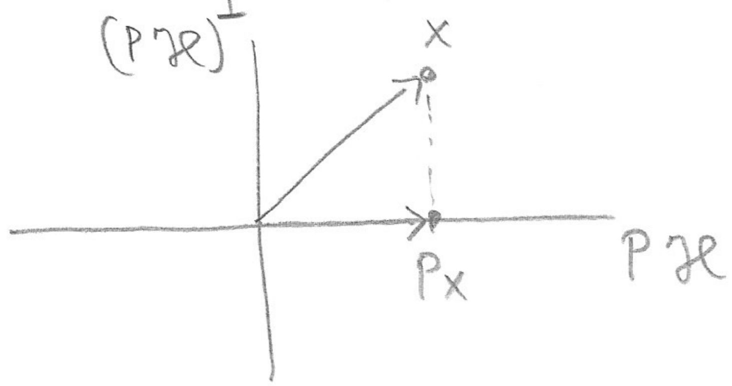
b) P is a (orthogonal) <u>projection</u>, if $P^2 = P = P^*$

c) U is <u>unitary</u>, if $U^*U = UU^* = 1$

d) V is an <u>isometry</u>, if $V^*V = 1$

e) N is <u>normal</u>, if $NN^* = N^*N$

2.11. Remarks: 1) P projects from $\mathcal{H}$ onto the subspace $P\mathcal{H}$:



2) $U \hat{=}$ rotation of coordinate system, maps an ONB to another ONB

3) V preserves the length:

$$\|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, \underbrace{V^*V}_{1} x\rangle$$

$$= \langle x, x\rangle = \|x\|^2$$

4) In finite dimensions, any isometry is unitary. In infinite dimensions this is not true any more; e.g.:

let $e_0, e_1, e_2, \ldots$ be an ONB of $\mathcal{H}$ and consider

$Ve_i = e_{i+1}$ one-sided shift

then $V^* e_i = \begin{cases} e_{i-1} & i \geq 1 \\ 0 & i = 0 \end{cases}$

V is isometry, but not unitary