3. Unbounded Operators

3.1. Motivation: We want to understand possible realizations of CCR

$$QP - PQ = i \cdot 1 \qquad (\text{put } \hbar = 1)$$

Immediate realizations are given on $L^2(\mathbb{R})$ by

$Q \hat{=}$ multiplication operator

$P \hat{=}$ differentiation operator

$$(Qf)(t) = t f(t)$$

$$(Pf)(t) = -i f'(t)$$

Then, formally,

$$(QP - PQ)f(t) = -i\Big(t \cdot f'(t) - \underbrace{(t \cdot f(t))'}_{t \cdot f'(t) + f(t)}\Big)$$

$$= +i f(t)$$

However, those P, Q are not bounded; actually there are no bounded operators satisfying CCR $\rightsquigarrow$ need for unbounded operators

3.2. Theorem: There are no $P, Q \in B(\mathcal{H})$ satisfying $QP - PQ = i \cdot 1$

[We do not need to require that P, Q are selfadjoint.]

Proof: Assume $QP - PQ = i \cdot 1$

Then we have for all $n \in \mathbb{N}$:

$$Q^n P - P Q^n = i n Q^{n-1} \neq 0 \qquad (*)$$

this follows by induction:

$n = 1$: ✓

$n \to n+1$: $Q^{n+1} P - P Q^{n+1} =$

$$= Q^n \underbrace{(QP - PQ)}_{i \cdot 1} + \underbrace{(Q^n P - P Q^n)}_{i \cdot n Q^{n-1}} Q$$

$$= i(n+1) \cdot Q^n$$

$\neq 0$ (otherwise: $Q^n P - P Q^n = 0$) ↯

From $(*)$ we get

$$n \| Q^{n-1} \| = \| Q^n P - P Q^n \|$$

$$\leq \underbrace{\| Q^n P \|}_{\leq \| Q^n \| \cdot \| P \|} + \underbrace{\| P Q^n \|}$$

$$= 2\,\|Q^n\|\cdot\|P\|$$
$$\leq 2\cdot\|Q^{n-1}\|\cdot\|Q\|\cdot\|P\|$$

Since $Q^{n-1}\neq 0$, we have $\|Q^{n-1}\|\neq 0$

$\Rightarrow n \leq 2\,\|Q\|\cdot\|P\| \quad \forall n\in\mathbb{N}$ ↯ □

3.3. Def.: A (unbounded) operator T on a Hilbert space $\mathcal{H}$ is given by

- a domain $D(T) \subset \mathcal{H}$, which is a linear subspace of $\mathcal{H}$
- a linear mapping $T: D(T)\to\mathcal{H}$

Usually we assume that $D(T)$ is dense in $\mathcal{H}$.

3.4. Example: Position operator Q

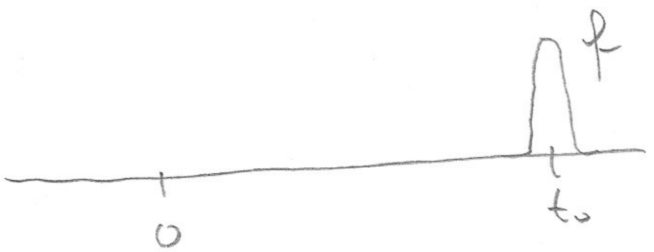
$$\mathcal{H} = L^2(\mathbb{R}) = \{f: \mathbb{R}\to\mathbb{C} \text{ measurable}, \int |f(t)|^2\,dt < \infty\}$$

$(Qf)(t) := t\,f(t)$ for all $t\in D(Q)$ where

$$D(Q) := \{f\in L^2(\mathbb{R}) \mid \int t^2\,|f(t)|^2\,dt < \infty\}$$

dense in $L^2(\mathbb{R})$ (but not equal)

Note that Q is unbounded (on its domain):



$Qf \approx t_0 \cdot f$

take such f with $\int |f(t)|^2 \, dt = 1$,

then $\int t^2 |f(t)|^2 \, dt \approx t_0^2 \cdot 1$, thus

$\dfrac{\|Qf\|}{\|f\|}$ can become arbitrarily large,

i.e. $\|Q\| = \infty$

<u>3.5. Remark:</u> We should choose $D(T)$ as large as possible.

1) If T is continuous on $D(T)$, we can extend definition from dense subset to $\mathcal{H}$.
Note that boundedness of T gives:
if $x_n \to x$, then $Tx_n \to y$
and this y is the same for all such sequences, so we can define $Tx := y$

2) If T is not bounded, we can still make the above extension, if we require the convergence of all $Tx_n \to y$ to same y.

This is a weaker kind of continuity property for our T, called "closable", and the extension of T gives a "closed" operator.

3) In order to control (x_n) and (Tx_n) together one works on the graph of the operator.

<u>3.6. Definition:</u> T_2 is an <u>extension</u> of T_1 if $D(T_1) \subset D(T_2)$ and

$$T_1 x = T_2 x \qquad \forall x \in D(T_1)$$

We write then: $T_1 \subset T_2$

<u>3.7. Def</u>: 1) For $T : D(T) \to \mathcal{H}$ its <u>graph</u> is

$$\Gamma(T) := \{(x, Tx) \mid x \in D(T)\} \subset \mathcal{H} \times \mathcal{H}.$$

2) T is <u>closed</u> if $\Gamma(T)$ is closed in $\mathcal{H} \times \mathcal{H}$

i.e.
$$\underbrace{\left.\begin{array}{l} (x_n, y_n) \to (x, y) \\ x_n, y_n \in \Gamma(T) \ \forall n \end{array}\right\}}_{\substack{x_n \in D(T),\ y_n = Tx_n \\ x_n \to x,\ Tx_n = y_n \to y}} \Rightarrow \underbrace{(x, y) \in \Gamma(T)}_{\substack{x \in D(T) \\ y = Tx}}$$

3) T is <u>closable</u> if it has a closed extension. It has then a smallest closed extension, called its <u>closure</u>, denoted by $\overline{T}$.

[Actually, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.]

<u>3.8. Def.:</u> Let $T : D(T) \to \mathcal{H}$ be densely defined. Put

$$D(T^*) := \{x \in \mathcal{H} \mid \exists y \in \mathcal{H} : \langle Tz, x\rangle = \langle z, y\rangle \quad \forall z \in D(T)\}$$

and define $T^* x := y$ (note that y uniquely determ.)

T^* is the <u>adjoint</u> of T.

<u>3.9. Remarks:</u> 1) We have defined T^* so that we have

$$\langle Tz, x\rangle = \langle z, T^* x\rangle \qquad \forall z \in D(T),\ x \in D(T^*)$$

3) $T_1 \subset T_2 \Rightarrow T_2^* \subset T_1^*$

2) $x \in D(T^*) \overset{\text{Riesz}}{\Longleftrightarrow} z \mapsto \langle Tz, x\rangle$ continuous
$D(T) \to \mathbb{C}$

4) $D(T^*)$ is clearly a linear subspace, but in general it does not need to be dense in $\mathcal{H}$!

3.10. Theorem: Let $T: D(T) \to \mathcal{H}$ be densely defined.

1) T closable $\iff$ $D(T^*)$ dense in $\mathcal{H}$

In this case we have: $\overline{T} = T^{**}$

2) T closable $\Rightarrow$ $(\overline{T})^* = T^*$

(in particular: T^* is always closed)

"Proof": Why is "being closable" related to $D(T^*)$?

Assume that T is not closable, i.e.

$$\begin{array}{lll} x_n \searrow & & T x_n \to y \\ & x, \quad \text{but} & \quad\quad \neq \\ \tilde{x}_n \nearrow & & T\tilde{x}_n \to \tilde{y} \end{array}$$

then: $\langle T(x_n - \tilde{x}_n), w\rangle = \langle x_n - \tilde{x}_n, T^* w\rangle$
$\forall w \in D(T^*)$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \langle y - \tilde{y}, w\rangle & & 0 \end{array}$$

$\Rightarrow \quad 0 \neq y - \tilde{y} \perp D(T^*)$

$\Rightarrow \quad D(T^*)$ is not dense in $\mathcal{H}$

Formal proof goes via graph:

$$V : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H} \qquad \text{is unitary}$$
$$(x, y) \mapsto (-y, x)$$

and one has

$$\Gamma(T^*) = V(\Gamma(T)^\perp)$$

and relies on the general facts for linear subspaces $\mathcal{M} \subset \mathcal{H}$:

$\mathcal{M}^\perp$ is always closed

$\mathcal{M} \subset (\mathcal{M}^\perp)^\perp$

"$=$" if $\mathcal{M}$ closed

$\square$

<u>3.11. Proposition:</u> If T is an unbounded operator, then we have

$$\underset{\substack{\uparrow \\ \text{range}}}{\operatorname{ran}}(T)^\perp = \underset{\substack{\uparrow \\ \text{kernel}}}{\ker}(T^*)$$

Note that kernel of closed operator is always closed, but this is not true in general for the range.

<u>Proof:</u> $x \in \operatorname{ran}(T)^{\perp}$

$\Leftrightarrow \langle x, Ty \rangle = 0 \quad \forall y \in D(T)$

$\Leftrightarrow x \in D(T^*)$ and $T^* x = 0$

$\Leftrightarrow x \in \ker(T^*)$ $\square$