4. Symmetric and selfadjoint operators

4.1. Def.: 1) T is called <u>symmetric</u> (or <u>Hermitean</u>), if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in D(T),$$

i.e., if $T \subset T^*$.

2) T is called <u>selfadjoint</u>, if

$T = T^*$, i.e.

T is symmetric and $D(T) = D(T^*)$

3) T is called <u>essentially selfadjoint</u>, if $\overline{T}$ is selfadjoint.

4.2. Remarks: 1) Note that a symmetric T is closable, since then

$$D(T^*) \supset \underbrace{D(T)}_{\text{dense in } \mathcal{H}}$$

2) T essentially selfadjoint

$\Longleftrightarrow$ T has exactly one s.a. extension (namely $\overline{T}$)

since: $S \subset T$, $S = S^*$, $T = T^*$
$\Rightarrow T = T^* \subset S^* = S \Rightarrow T \subset S$ $\Big\} \Rightarrow T = S$

3) $\{T$ essentially s.a.$\} \subsetneqq \{T$ symmetric$\}$

4) All the good things (spectral theorem, Theorem of Stone) are only true for (essentially) s.a. operators, but not for symmetric operators in general.

<u>4.3. Theorem</u> (basic criterion for selfadjointness):
Let $T: D(T) \to \mathcal{H}$ be symmetric. TFAE:

i) T is selfadjoint

ii) T is closed and $\ker(T^* + i) = \{0\}$,
and $\ker(T^* - i) = \{0\}$

iii) $\operatorname{ran}(T - i) = \mathcal{H}$ and
$\operatorname{ran}(T + i) = \mathcal{H}$

[Note $T+i$ means $T + \underset{\sqrt{-1}}{i} \cdot id$ and
$D(T+i) = D(T)$]

<u>Proof</u>: i) $\Rightarrow$ ii) $T = T^*$ closed

Consider $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (e.g., $\lambda = \pm i$)

we have to show that $\ker(T^* - \lambda) = 0$

Let $x \in D(T^* - \lambda) = D(T^*) = D(T)$

s.th. $\underbrace{(T^* - \lambda)}_{(T - \lambda)} x = 0$ i.e. $Tx = \lambda x$, $Tx = T^* x$

but then

$$\bar{\lambda} \langle x, x \rangle = \langle \lambda x, x \rangle = \langle T^* x, x \rangle$$
$$= \langle x, Tx \rangle$$
$$= \langle x, \lambda x \rangle = \lambda \langle x, x \rangle$$

Since $\lambda \neq \bar{\lambda} \neq 0 \Rightarrow \langle x, x \rangle = 0$

$$\Rightarrow x = 0$$

thus: $\ker(T^* - \lambda) = \{0\}$

ii) $\Rightarrow$ iii) Assume $\ker(T^* + i) = \{0\}$

$\overset{3.11}{\Rightarrow}$ $\operatorname{ran}(T - i)^\perp = \ker(\underbrace{(T - i)^*}_{T^* + i}) = \{0\}$

$\Rightarrow \operatorname{ran}(T - i)$ dense in $\mathcal{H}$

still to show: $\operatorname{ran}(T - i)$ closed

Assume $y_n \in \operatorname{ran}(T-i)$ and $y_n \to y \in \mathcal{H}$

to show: $y \in \operatorname{ran}(T-i)$

Let $y_n = (T-i)x_n$

note that $\forall z \in D(T-i) = D(T)$

$$\|(T-i)z\|^2 = \langle (T-i)z, (T-i)z \rangle$$

$$= \langle Tz, Tz \rangle + \underbrace{i\langle z, Tz \rangle - i\underbrace{\langle Tz, z \rangle}_{\langle z, Tz \rangle}}_{=0} + \langle z, z \rangle$$

$$= \|Tz\|^2 + \|z\|^2$$

Now let $y_n = (T-i)x_n \to y$, thus (y_n) Cauchy sequence and

$$\|y_n - y_m\|^2 = \|(T-i)(x_n - x_m)\|^2$$

$$= \|T(x_n - x_m)\|^2 + \|x_n - x_m\|^2$$

$\Rightarrow$ (Tx_n) is Cauchy sequence in $\mathcal{H}$

and (x_n) is — " —

thus $\exists z \in \mathcal{H}: Tx_n \to z$

$\exists x \in \mathcal{H}: x_n \to x$

T closed $\Rightarrow$ $x \in D(T)$ and $Tx = z$

But then $y = \lim \underbrace{(T-i)x_n}_{\underbrace{Tx_n}_{\to z = Tx} - i \underbrace{x_n}_{\to x}}$

$$= (T-i)x \in \operatorname{ran}(T-i)$$

iii) ⇒ i) similar □

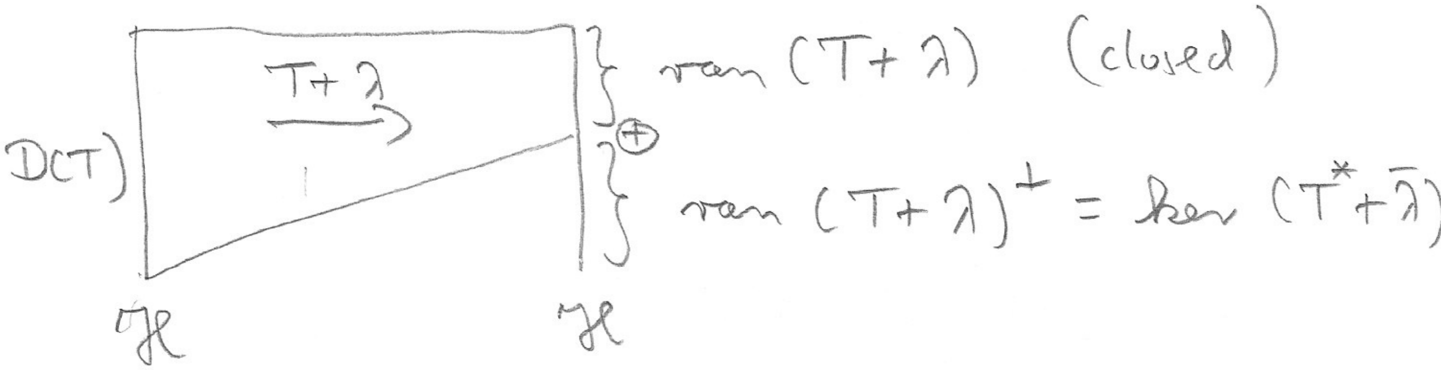
<u>4.4. Theorem:</u> Let $T : D(T) \to \mathcal{H}$ be symmetric. TFAE

i) T is essentially s.a.

ii) $\ker(T^* + i) = \{0\}$ and
$\ker(T^* - i) = \{0\}$

iii) $\operatorname{ran}(T-i)$ and $\operatorname{ran}(T+i)$ are dense in $\mathcal{H}$

<u>4.5. Remark:</u> 1) For a closed symmetric T and $\lambda = \pm i$ we have thus the situation



and we have

$$T \text{ s.a.} \iff \begin{cases} \dim \ker (T^* + \bar{\lambda}) = 0 \\ \qquad \text{both for } \lambda = +i \text{ and } \lambda = -i \end{cases}$$

$\lambda = \pm i$ might look special, but we have

$$\lambda \mapsto \dim \ker (T^* - \lambda)$$

is constant on upper half plane $\mathbb{C}^+$
and — " — lower — " — $\mathbb{C}^-$

2) The possible s.a. extensions of a symmetric operator T are completely characterized by its <u>defect indices</u>

$$m := \dim \ker (T^* - i) = \dim \operatorname{ran} (T + i)^\perp$$
$$n := \dim \ker (T^* + i) = \dim \operatorname{ran} (T - i)^\perp$$

In particular:

T is essentially s.a $\iff (m, n) = (0, 0)$

T has s.a. extensions $\iff \quad = (k, k)$

T has no s.a. extension $\iff \quad = (k, l)$
with $k \neq l$

4.6. Example: $T = Q$ position operator

$\mathcal{H} = L^2(I)$ where $I = [a,b], [a,\infty), (-\infty, b), (-\infty, \infty)$

$T : D(T) \to \mathcal{H}$ with

$(Tf)(t) = t \cdot f(t)$

$$D(T) = \{ f \in L^2(I) \mid \underbrace{t f(t) \in L^2(I)}_{\text{i.e. } \int_I t^2 |f(t)|^2 \, dt < \infty} \}$$

What is T^* ?

$$D(T^*) = \{ g \in \mathcal{H} \mid \exists h \in \mathcal{H} : \forall f \in D(T) \quad \langle Tf, g \rangle = \langle f, h \rangle \}$$

i.e. $\langle Tf, g \rangle = \langle f, h \rangle$

$$\langle Tf, g \rangle = \int \overline{t f(t)} \, g(t) \, dt \qquad \langle f, h \rangle = \int \overline{f(t)} \, h(t) \, dt$$

$$\Rightarrow \int \bar{f}(t) \underbrace{[t g(t) - h(t)]} \, dt = 0 \quad \forall f \in D(T)$$

$$\Rightarrow \quad " = 0 " \quad \text{a.e.}$$

$$\Rightarrow\ t\,g(t) = \underset{\displaystyle (T^*g)(t)}{\underset{\shortparallel}{h(t)}} \in \mathcal{H} \Rightarrow g \in D(T)$$

thus: $T = T^*$, i.e. T is selfadjoint

Note that here the condition $(m,n) = (0,0)$ is obvious!

dim rang $(T+i)^\perp$

dim $(T-i)^\perp$

4.7. Example: $T = P$ (momentum operator)

$T = i\dfrac{d}{dt}$, $Tf = if'$

Consider $\mathcal{H} = L^2([0,1])$ and put

$$D(T) := \{ f \in L^2 \mid f \text{ continuously diff'able},\ f(0) = 0 = f(1) \}$$

1) T is symmetric: we have for $f, g \in D(T)$

$$\langle Tf, g\rangle = \int_0^1 i f'(t)\, \overline{g(t)}\, dt$$

$$= -i \int_0^1 \bar{f}'(t)\, g(t)\, dt$$

partial integration

$$= i \int_0^1 \bar{f}(t)\, g'(t)\, dt \; - i \underbrace{[\bar{f} g]_0^1}_{\underbrace{\bar{f}(1)}_{=0} g(1) - \underbrace{\bar{f}(0)}_{=0} g(0) = 0}$$

$$= \int_0^1 \bar{f}(t)\, i g'(t)\, dt$$

$$= \langle f, Tg \rangle$$

thus $T \subset T^*$

2) What is T^*?

Let $g \in D(T^*)$, i.e. $\exists\, h = T^* g$ s.th.

$$\langle g, Tf \rangle = \langle h, f \rangle \quad \forall f \in D(T)$$

i.e.

$$\int \overline{g(t)}\, i f'(t)\, dt = \int \bar{h}(t)\, f(t)\, dt$$

write $h = H'$ a.e. by putting $H(t) := \int_0^t h(s)\, ds$

then:

$$\int \bar{g}(t)\, i f'(t)\, dt = \int \bar{H}'(t) f(t)\, dt$$

$$= - \int \bar{H}(t) f'(t)\, dt + \bar{H}(1)\underbrace{f(1)}_{=0} - \bar{H}(0)\underbrace{f(0)}_{=0}$$

$$\Rightarrow \int f'(t)\left[i \bar{g}(t) + \bar{H}(t) \right] dt = 0 \quad \forall f \in D(T)$$

$$\Rightarrow i\bar{g} + \bar{H} \in \{ f' \mid f \in D(T) \}^{\perp} = \mathbb{C} \cdot 1$$

Assignment

$$\Rightarrow \bar{g} = i\bar{H} + \text{const}$$

$$\Rightarrow g = \underbrace{-iH + \text{const}}_{\text{absolutely continuous (AC) function}}$$

note: no boundary condition for g and

$T^* g = h = H' = i g'$ (AC functions are a.e. diff'ble)

Thus

$D(T^*) \subset \{ g \in L^2 \mid g \text{ AC}, g' \in L^2 \}$

"=" by direct checking!

thus: $T \neq T^*$ and T is not selfadjoint

3) T is closable (as symmetric operator), but not closed; we get its closure as

$\overline{T} = T^{**}$ with domain

$$D(\overline{T}) = \{ f \in L^2 \mid f\ AC, f' \in L^2, \quad f(0) = 0 = f(1) \}$$

and $\overline{T} f = i f'$

So in particular, T^* is not symmetric, since $T^{**} \subsetneq T^*$

and we have

$$\overline{T} = T^{**} \subsetneq T^* = \overline{T}^*$$

thus $\overline{T}$ is not selfadjoint (and T not essentially selfadjoint)

Closure moves us to the right class of functions ("we want diff'ble fcts") but the problem with selfadjointness lies in boundary conditions and closure cannot tell us how to deal with this ☹

4) What are defect indices of T?

$m = \dim \ker (T^* - i)$

$$\underbrace{T^* f}_{i f'} = i f \implies f = \alpha e^{t}$$

$\implies m = 1$

also: $n = 1$ (with $T^* f = -i f$ for $f = \alpha e^{-t}$)

$\implies (m, n) = (1, 1)$

i.e. T has s.a. extensions

5) What are the s.a. extensions S?

Note problem with boundary conditions:

$$\langle f, S g \rangle = \langle S f, g \rangle$$

$$i \int \bar{f} g' \qquad\qquad -i \int \overline{f'} g$$

$$= i \int \bar{f} g' - i \underbrace{\left[\overline{f(1)} g(1) - \overline{f(0)} g(0) \right]}_{\overset{!}{=} 0}$$

boundary conditions for f must be so that they enforce the same conditions for g

$f(0) = 0 = f(1) \iff$ no conditions for g

$f(1) = \alpha \cdot f(0) \iff g(1) = \alpha g(0)$

for some $\alpha \in \mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$

then: $\underbrace{\overline{f(1)}}_{\bar{\alpha}\overline{f(0)}}\underbrace{g(1)}_{\alpha g(0)} - \overline{f(0)}g(0) = 0$

So we each $\alpha \in \mathbb{T}$ we put

$D(T_\alpha) := \{f \in L^2 \mid f \text{ AC}, f' \in L^2,$
$f(1) = \alpha f(0)\}$

$T_\alpha f = i f'$

Then all those T_α are (different) s.a. extensions of T. And that's all possible extensions

T^* not symmetric

$\cup$

T_α s.a. extension of T (for each α with $|\alpha|=1$)

$\cup$

T symmetric, but not essentially s.a.