5. Spectrum and Spectral Theorem

Consider $\dim \mathcal{H} < \infty$, then

$A \in B(\mathcal{H}) \mathrel{\hat{=}}$ matrix

spectrum of A $\hat{=}$ eigenvalues of A
$\sigma(A)$ ($\hat{=}$ "possible values of measurements for A")

spectral theorem $\hat{=}$ diagonalisation of A
(for s.a. A)

Goal: generalize this to $\dim \mathcal{H} = \infty$
and unbounded operators !

$$\begin{aligned}
\lambda \in \sigma(A) &\iff \lambda \text{ eigenvalue of } A \\
&\iff \exists\, x \neq 0 : Ax = \lambda x \\
&\iff \exists\, x \neq 0 : (A - \lambda)x = 0 \\
&\overset{\dim \mathcal{H} < \infty}{\iff} A - \lambda \text{ not invertible}
\end{aligned}$$

note: $\dim \mathcal{H} < \infty \Rightarrow$ injective = surjective
= bijective

"correct" generalization of "eigenvalue" λ to infinite dimensions: $(A-\lambda)^{-1}$ does not exist !

<u>5.1. Def.</u>: 1) Let $T : D(T) \to \mathcal{H}$ be a closed operator. The <u>resolvent set</u> of T is

$$\rho(T) := \{ \lambda \in \mathbb{C} \mid T - \lambda : D(T) \to \mathcal{H} \text{ is a bijection and } (T-\lambda)^{-1} \in B(\mathcal{H}) \};$$

its complement

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

is the <u>spectrum</u> of T

2) The spectrum $\sigma(T)$ can be written as a disjoint union

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

where

$$\sigma_p(T) := \{ \lambda \in \mathbb{C} \mid \exists x \neq 0 : Tx = \lambda x \}$$

<u>point spectrum</u> ("eigenvalues")

$$\sigma_c(T) := \{ \lambda \in \mathbb{C} \mid \lambda \notin \sigma_p(T), \operatorname{ran}(T-\lambda) \text{ dense in } \mathcal{H}, \text{ but } (T-\lambda)^{-1} : \operatorname{ran}(T-\lambda) \to D(T-\lambda) = D(T) \text{ is not bounded} \}$$

<u>continuous spectrum</u>

$$\sigma_r(T) := \{\lambda \in \mathbb{C} \mid \lambda \notin \sigma_p(T),\ \mathrm{ran}(T-\lambda) \text{ is not dense in } \mathcal{H}\}$$

residual spectrum

5.2 Remark: 1) The point spectrum consists of eigenvalues, for which there exists eigenvectors, i.e.

$$Tx = \lambda x \qquad \text{for some } x \neq 0$$

2) The continuous spectrum consists of λ for which we have approximate eigenvectors: Let $\lambda \in \sigma_c(T)$, then

$$T-\lambda : D(T) \to \underbrace{\mathrm{ran}(T-\lambda)}_{\text{dense in } \mathcal{H}} \quad \text{bijective}$$

but: $(T-\lambda)^{-1} : \mathrm{ran}(T-\lambda) \to D(T)$

is, by assumption, not bounded, i.e. $\exists\, x_n \in \mathrm{ran}(T-\lambda)$ with $\|x_n\| = 1$, but $\|(T-\lambda)^{-1} x_n\| =: d_n \xrightarrow{n\to\infty} \infty$

Then put $y_n := \frac{1}{d_n}(T-\lambda)^{-1} x_n$

$\Rightarrow \|y_n\| = 1$ and $(T-\lambda) y_n = \frac{1}{d_n} x_n$

thus: $\|(T-\lambda) y_n\| = \frac{1}{d_n}\|x_n\| \xrightarrow{n\to\infty} 0$

(y_n) are approximate eigenvectors, $Ty_n \approx \lambda y_n$

3) In many cases the residual spectrum is absent; in particular:

T is selfadjoint $\Rightarrow \sigma_r(T) = \emptyset$

4) For the spectrum of a closed symmetric operator there are the following possibilities:

i) $\sigma(T) = \mathbb{C}_0^+$

ii) $\sigma(T) = \mathbb{C}_0^-$

iii) $\sigma(T) = \mathbb{C}$

iv) $\sigma(T) \subseteq \mathbb{R}$

and

$\sigma(T) \subseteq \mathbb{R} \Leftrightarrow T$ is selfadjoint

5.3. Example: position (multiplication) operator

see 4.6. $I = [a,b], [a,\infty), (-\infty,b), (-\infty,\infty)$

$\mathcal{H} = L^2(I)$ $(Tf)(t) = t\,f(t)$ on

$D(T) = \{ f \in L^2(I) \mid Tf \in L^2(I) \}$

By 4.6 we know that $T = T^*$, thus $\sigma(T) \subseteq \mathbb{R}$

Look first for eigenvalues $\lambda \in \mathbb{R}$:

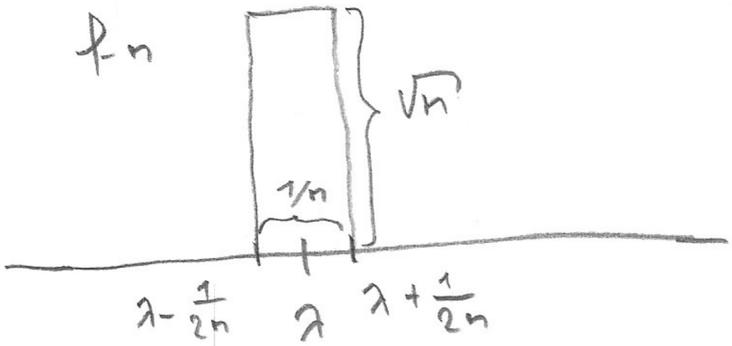
$Tf = \lambda f$, i.e. $t f(t) = \lambda f(t)$ a.e.

has no non-trivial solution in $\mathcal{H}$;

thus $\sigma_p(T) = \phi$

but we can approximate the "δ-function" at λ with actual elements in $\mathcal{H}$, if $\lambda \in I$

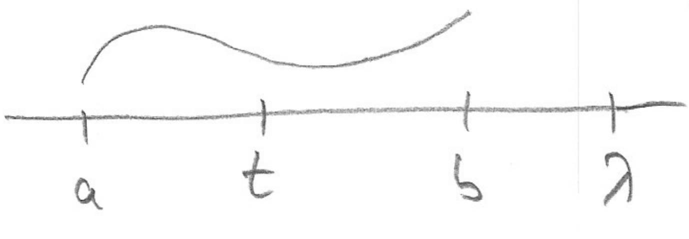
f_n



thus: $\|f_n\| = 1$ and $T f_n \approx \lambda f_n$

$\Rightarrow I \ni \lambda \in \sigma(T)$

If $\lambda \notin I$, then $\lambda \in \rho(T)$



$$(T - \lambda)^{-1} f(t) = \frac{f(t)}{t - \lambda} \in L^2([a,b])$$

for $\lambda \notin [a,b]$

thus: $\sigma_p(T) = \phi$

$\sigma(T) = \sigma_c(T) = I \subseteq \mathbb{R}$

5.4. Motivation: 1) Consider $\dim \mathcal{H} =: n < \infty$ and $A \in B(\mathcal{H})$ selfadjoint (corresponding to s.a. matrix). Write

$\sigma(A) = \{\lambda_1 < \lambda_2 < \dots < \lambda_k\}$, where $\lambda_i \in \mathbb{R}$

and let, for $\lambda \in \sigma(A)$,

$$\mathcal{H}_\lambda := \{x \in \mathcal{H} \mid Ax = \lambda x\}$$

be eigenspace to eigenvalue λ.

Then spectral theorem for s.a. matrices can be stated as:

$$\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathcal{H}_\lambda \quad , \text{ i.e.}$$

- $\mathcal{H}_{\lambda_i} \perp \mathcal{H}_{\lambda_j}$ for $i \neq j$
- each $x \in \mathcal{H}$ can uniquely be written as

$$x = \sum_{i=1}^{k} x_i \quad \text{with } x_i \in \mathcal{H}_{\lambda_i}$$

If we denote by

$$P_\lambda : \mathcal{H} \to \mathcal{H}_\lambda \qquad (\lambda \in \sigma(A))$$

the orthogonal projection onto the eigenspace for λ, then

$$x_i = P_{\lambda_i} x$$

and we have for $x \in \mathcal{H}$:

$$Ax = \sum_i A x_i = \sum_i \lambda_i x_i = \sum \lambda_i P_{\lambda_i} x,$$

thus

$$A = \sum_{i=1}^{k} \lambda_i P_{\lambda_i} = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$$

2) We like to generalize this to infinite dimensions!

So consider $\dim \mathcal{H} = \infty$ and $T = T^*$,

then $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$

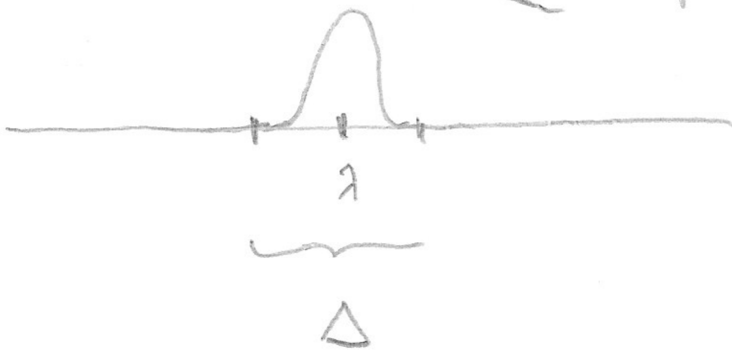
this works as above

here $\mathcal{H}_\lambda = \{0\}$ and $P_\lambda = 0$

however: we can replace this by approximate versions

Consider multiplication operator Q on $L^2(\mathbb{R})$; we have no functions localized exactly at λ, but we can localize them in (small) intervals Δ

$\leftarrow f \in \mathcal{H}_\Delta$



$$\mathcal{H}_\Delta := \{ f \in L^2(\mathbb{R}) \mid f(t) = 0 \ \forall t \notin I \}$$

$E_\Delta : \mathcal{H} \to \mathcal{H}_\Delta$ orth. projection

then we have

- for $\Delta_1 \cap \Delta_2 = \emptyset$: $\mathcal{H}_{\Delta_1} \perp \mathcal{H}_{\Delta_2}$

- $\Delta = \Delta_1 \dot\cup \Delta_2 \Rightarrow E_\Delta = E_{\Delta_1} + E_{\Delta_2}$

  in particular: $\mathcal{H}_\Delta = \mathcal{H}_{\Delta_1} \oplus \mathcal{H}_{\Delta_2}$

for $\mathbb{R} = \dot\bigcup_i \Delta_i$: $\mathcal{H} = \bigoplus_i \mathcal{H}_{\Delta_i}$

How can we represent Q ?

$$\left.\begin{array}{l}|\Delta| \text{ small} \\ x \in \mathcal{H}_\Delta\end{array}\right\} \Rightarrow Qx \approx \lambda x \quad (\lambda \in \Delta)$$

thus: $Qx = \sum_i \underbrace{Q E_{\Delta_i} x}_{\approx \lambda_i E_{\Delta_i} x \quad (\lambda_i \in \Delta_i)}$

i.e. $Q \hat{=} \lim_{|\Delta_i| \to 0} \sum_i \lambda_i E_{\Delta_i} = \int \lambda \, dE(\lambda)$

with $E(\lambda) := E_{(-\infty, \lambda]}$

This is true in general for selfadjoint T, it can be written as an operator-valued Stieltjes integral:

$$T = \int \lambda \, dE(\lambda)$$

for a corresponding "resolution of identity" $(E(\lambda))_{\lambda \in \mathbb{R}}$

We also write E_λ for $E(\lambda)$.

<u>5.5. Def.</u>: A family $(E(\lambda))_{\lambda\in\mathbb{R}} \subset B(\mathcal{H})$ is called <u>resolution of the identity</u>, or <u>projection-valued measure</u> (PVM), if we have:

i) for each $\lambda \in \mathbb{R}$, E_λ is orthogonal projection, i.e. $E_\lambda^* = E_\lambda = E_\lambda^2$

ii) for all $\lambda \le \mu$,

$$E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$$

iii)

$$\lim_{\lambda \to -\infty} E_\lambda x = 0$$

$$\lim_{\lambda \to +\infty} E_\lambda x = x \qquad \forall x \in \mathcal{H}$$

iv) $\lambda \mapsto E_\lambda$ is right continuous, i.e.

$$\lim_{\varepsilon \searrow 0} E_{\lambda+\varepsilon} x = E_\lambda x \qquad \forall x \in \mathcal{H}$$

<u>5.6. Remark</u>: 1) Note that the properties of (E_λ) ensure that, for each $x \in \mathcal{H}$,

$$\lambda \mapsto \langle x, E_\lambda x \rangle, \quad \mathbb{R} \to \mathbb{R}$$

has precisely the properties of a distribution fct, i.e. we can define Lebesgue-Stieltjes integral $\int f(\lambda)\, d\langle x, E_\lambda x\rangle$

2) In order to define $\int f(\lambda)\, dE_\lambda$ as an operator we need to define

$\langle x, \int f(\lambda)\, dE_\lambda y \rangle$; by polarization it suffices to define this for $x = y$, but then we can define

$$\langle x, \int f(\lambda)\, dE_\lambda x \rangle := \underbrace{\int f(\lambda)\, d\langle x, E_\lambda x\rangle}_{\text{this is an ordinary Stieltjes integral}}$$

<u>5.7. Proposition:</u> Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a projection valued measure and $f: \mathbb{R} \to \mathbb{C}$ measurable. Then there exists a densely defined operator

$$T_f = \int f(\lambda)\, dE_\lambda$$

with domain

$$D(T_f) = \{x \in \mathcal{H} \mid \int |f(\lambda)|^2\, d\langle x, E_\lambda x\rangle < \infty\}$$

and uniquely determined by

$$\langle x, T_f x\rangle = \int f(\lambda)\, d\langle x, E_\lambda x\rangle \quad \forall x \in D(T_f)$$

Furthermore we have for all $x \in D(T_f)$

$$\|T_f x\|^2 = \int |f(\lambda)|^2\, d\langle x, E_\lambda x\rangle$$

5.8. Spectral Theorem: 1) Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a PVM. Then

$$T := \int \lambda \, dE_\lambda$$

is a selfadjoint operator, with domain

$$D(T) = \{x \in \mathcal{H} \mid \int \lambda^2 \, d\langle x, E_\lambda x\rangle < \infty\}$$

2) Let T be an unbounded selfadjoint operator. Then there exists a uniquely determined resolution of identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that

$$T = \int \lambda \, dE_\lambda .$$

5.9 Examples: 1) $\dim \mathcal{H} < \infty$

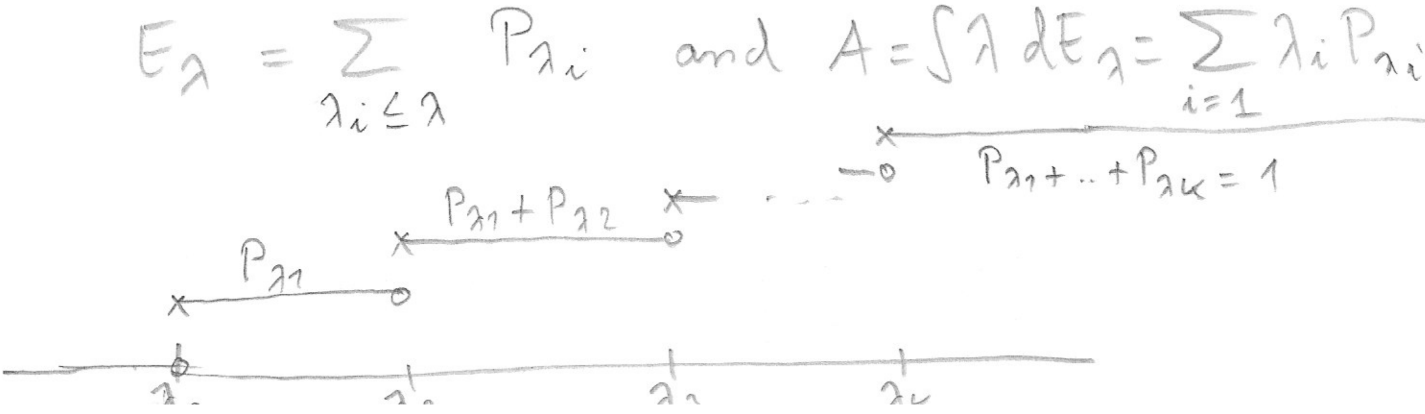
$\sigma(A) = \{\lambda_1 < \dots < \lambda_k\}$

P_λ orthogonal projection onto

$\mathcal{H}_\lambda := \{x \in \mathcal{H} \mid Ax = \lambda x\}$ for $\lambda \in \sigma(A)$

Then

$$E_\lambda = \sum_{\lambda_i \le \lambda} P_{\lambda_i} \quad \text{and} \quad A = \int \lambda \, dE_\lambda = \sum_{i=1}^{k} \lambda_i P_{\lambda_i}$$

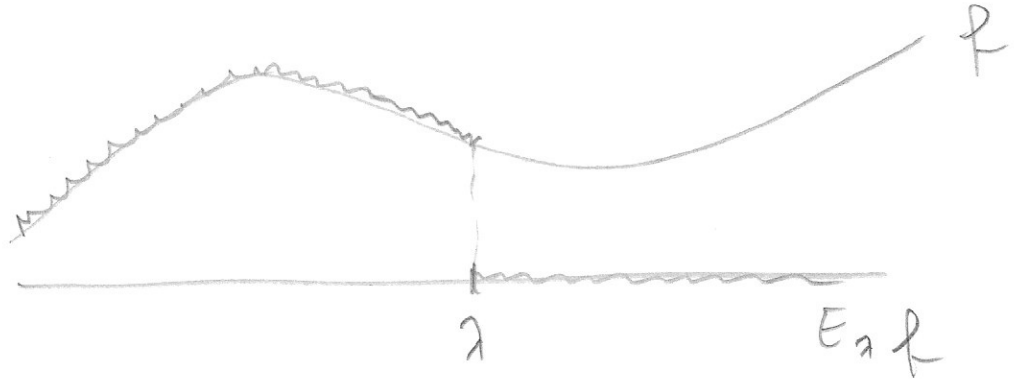


2) Multiplication operator

$(Qf)(t) = t\, f(t)$ on $\mathcal{H} = L^2(\mathbb{R})$

Then the resolution of identity $(E_\lambda)_{\lambda \in \mathbb{R}}$ is given by

$$(E_\lambda f)(t) = \begin{cases} f(t) & t \leq \lambda \\ 0 & t > 0 \end{cases}$$



in this case E_λ increase continuously in λ

$$Q = \int \lambda \, dE(\lambda)$$