5.10. Theorem (Functional Calculus):

Let T be a s.a. operator with spectral decomposition

$$T = \int \lambda \, dE(\lambda).$$

1) For each measurable function $f : \mathbb{R} \to \mathbb{C}$ there exists an operator

$$f(T) := \int f(\lambda) \, dE(\lambda), \text{ i.e.}$$

$$D(f(T)) = \{x \in \mathcal{H} \mid \int |f(\lambda)|^2 \, d\langle x, E_\lambda x\rangle < \infty\}$$

and

$$\langle x, f(T)x\rangle = \int f(\lambda) \, d\langle x, E_\lambda x\rangle$$

$$\forall \, x \in D(f(T))$$

2) Let $f, g : \mathbb{R} \to \mathbb{C}$ be measurable.

Then $f(T) g(T) \subset (fg)(T)$ and

$$D(f(T)g(T)) = D(g(T)) \cap D((fg)(T))$$

3) For measurable $f : \mathbb{R} \to \mathbb{C}$ we have $f(T)^* = \bar{f}(T)$. In particular:

f real-valued $\Rightarrow$ $f(T)$ selfadjoint

5.11. Remarks: 1) Note that this gives a very general functional calculus. For unbounded operators, not even polynomials are a priori well-defined, since

$$T^2 x = T(Tx)$$

might not have a dense domain.

2) For $f = 1_B$ for Borel sets $B \subset \mathbb{R}$ we get then our "projection-valued measure" μ_T:

$$B \mapsto \mu_T(B) = 1_B(T) = \int_B dE(\lambda)$$

$$= E(B)$$

with the properties:

- $E(B)$ is an orthogonal projection for each B
- $E(\emptyset) = 0$, $E(\mathbb{R}) = 1$
- $E(B_1)E(B_2) = E(B_1 \cap B_2)$
- $B = \dot{\bigcup}_n B_n \Rightarrow E(B)x = \sum_n E(B_n)x \quad \forall x \in \mathcal{H}$

3) Note that for $B_i \cap B_j = \phi \; (i \neq j)$

we have

$$\| \sum \alpha_i E(B_i) \| = \max_{i} |\alpha_i|$$

for which

$E(B_i) \neq 0$

and in the same way for the integrals

$$\| \int f(\lambda)\, dE(\lambda) \| = \operatorname*{ess\,sup}_{\lambda \in \underbrace{supp(E)}} |f(\lambda)|$$

smallest closed set G
such that $E(G^c) = 0$

If $f(\lambda) = (\lambda - \lambda_0)^{-1}$ for fixed $\lambda_0 \in \mathbb{R}$

then

$$(T - \lambda_0)^{-1} = \int \frac{1}{\lambda - \lambda_0}\, dE(\lambda) \quad \text{and}$$

$$\| (T - \lambda_0)^{-1} \| = \operatorname*{ess\,sup}_{\lambda \in supp(E)} \left| \frac{1}{\lambda - \lambda_0} \right|$$

$$\| \qquad \begin{cases} < \infty & \text{if } \lambda_0 \in \rho(T) \\ = \infty & \text{if } \lambda_0 \in \sigma(T) \end{cases}$$

Thus $supp(E) = \sigma(T)$ and

$$\int f\, dE(\lambda) = \int_{\sigma(T)} f\, dE(\lambda)$$

f only needs to be defined on $\sigma(T)$

5.12. von Neumanns axiomatic of quantum mechanics

1) A quantum mechanical system corresponds to a complex separable Hilbert space $\mathcal{H}$.

2) The pure states correspond to unit vectors $x \in \mathcal{H}$ ($\|x\| = 1$). Actually, αx, for $\alpha \in \mathbb{C}$, $|\alpha| = 1$, describes the same state as x.

[Pure states are the ones where one has total knowledge, general states contain an additional statistical component - which we don't consider here.]

3) Observables correspond to selfadjoint operators on $\mathcal{H}$.

4) The measurement of an observable corresponding to $T = \int \lambda \, dE(\lambda)$ on a system corresponding to $x \in \mathcal{H}$ has as possible outcome values $\lambda \in \sigma(T)$

with a probability distribution given by $d\langle x, E(\lambda)x\rangle$, i.e.

$$\text{prob}(\lambda \in (-\infty, \lambda_0]) = \langle x, E(\lambda_0)x\rangle$$

or for any Borel set $B \subset \mathbb{R}$

$$\text{prob}(\lambda \in B) = \langle x, E(B)x\rangle$$

5) In the setting of 4) and for a measurable function $f: \mathbb{R} \to \mathbb{R}$, the s.a. operator $f(T)$ corresponds to the function f(observable).

<u>5.13. Remarks:</u> 1) Let $\lambda \in \sigma_p(T)$, then $E(\{\lambda\}) \neq 0$ is projection onto vectors which give value λ with certainty

2) If $\lambda \in \sigma_c(T)$ then $E(\{\lambda\}) = 0$, so probability of measuring exactly λ is zero, but for each interval $(\varepsilon > 0)$ $\Delta_\varepsilon := (\lambda - \varepsilon, \lambda + \varepsilon)$, $E(\Delta_\varepsilon) \neq 0$, thus we have states $x_\varepsilon \in \mathcal{H}$ s.th. measurement in such case gives with probability 1 a result in Δ_ε.